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HOMOCLINIC ORBITS, SUBHARMONICS AND GLOBAL BIFURCATIONS  
IN FORCED OSCILLATIONS

Bernie Greenspan and Philip Holmes

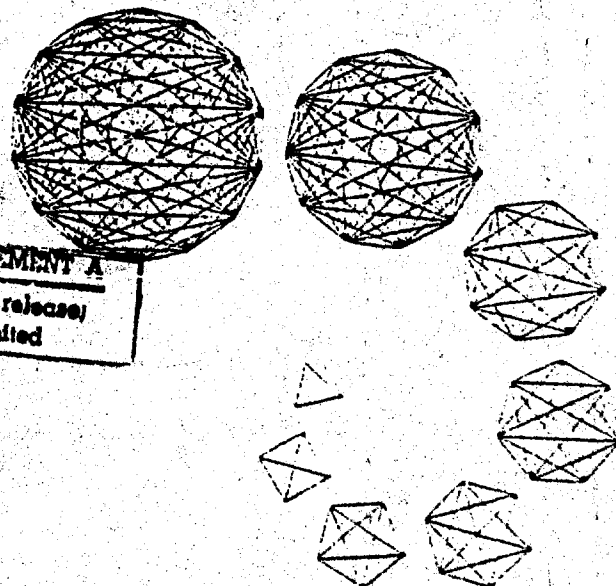
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B.D. Greenspan and P.J. Holmes

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HOMOCLINIC ORBITS, SUBHARMONICS AND GLOBAL BIFURCATIONS  
IN FORCED OSCILLATIONS

by

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ABSTRACT

We develop a method for the analysis of perturbations of integrable planar systems of differential equations. Concentrating on the case in which the unperturbed system is Hamiltonian and the perturbation introduces dissipation and time-periodic forcing, we show how the global solution curves (level sets) of the unperturbed system can be used in regular perturbation calculations to locate subharmonic orbits and homoclinic orbits and to characterize the bifurcations in which they are created as external parameters are varied. We apply our results to Duffing's equation and point out applications to the chaotic motions of buckled elastic beams undergoing periodic excitation.

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## 1. INTRODUCTION: A PHYSICAL EXAMPLE

In this article we develop a method for the analysis of systems of the form

$$\dot{x} = f(x) + \varepsilon g(x;t), \quad (1.1)$$

where  $x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$ , and  $f = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}$ ,  $g = \begin{pmatrix} g_1(u,v;t) \\ g_2(u,v;t) \end{pmatrix}$  are sufficiently smooth (bounded) functions and  $g$  is  $T$ -periodic in  $t$ . In examples of interest  $g$  will generally also contain parameters

$$g_\mu(x,t) = g(x;t;\mu) \quad \mu \in \mathbb{R}^m. \quad (1.2)$$

The unperturbed system,

$$\dot{x} = f(x), \quad (1.3)$$

is assumed to be integrable. In particular, we shall concentrate on the Hamiltonian case, in which a real valued function  $H: \mathbb{R}^2 \rightarrow \mathbb{R}$  exists and

$$f_1(u,v) = \frac{\partial H}{\partial v}, \quad f_2(u,v) = -\frac{\partial H}{\partial u}. \quad (1.4)$$

The results we describe here can be generalized to multidimensional systems and even to infinite dimensional Hamiltonian systems arising on evolution equations in studies of partial differential equations (cf. Holmes and Marsden [1981a]), and to the study of multidegree of freedom Hamiltonian systems with perturbations which do not depend explicitly on time (cf. Churchill [1980], Holmes and Marsden [1981b,c]).

Our basic assumption, made precise in Section 3, is that the unperturbed system (1.3) possesses a nondegenerate (hyperbolic) saddle point  $p_0$  with a homoclinic orbit  $\Gamma^0 = \{g^0(t) | t \in \mathbb{R}\}$  such that

$$\lim_{t \rightarrow +\infty} q^0(t) = \lim_{t \rightarrow -\infty} q^0(t) = p_0. \quad (1.5)$$

We further assume that the interior of  $\Gamma^0$  is filled with a one-parameter family of periodic orbits  $\Gamma^\alpha = \{q^\alpha(t) | \alpha \in (-1, 0), t \in [0, T_\alpha]\}$  where  $T_\alpha$  is the period of  $q^\alpha(t)$ ,

$$\lim_{\alpha \rightarrow 0} q^\alpha(t) = q^0(t), \quad (1.6)$$

and  $\lim_{\alpha \rightarrow -1} q^\alpha(t) = q^{-1}$  is an elliptic fixed point or center, cf. Figure 1.1. The homoclinic orbit can be replaced by a homoclinic cycle connecting several

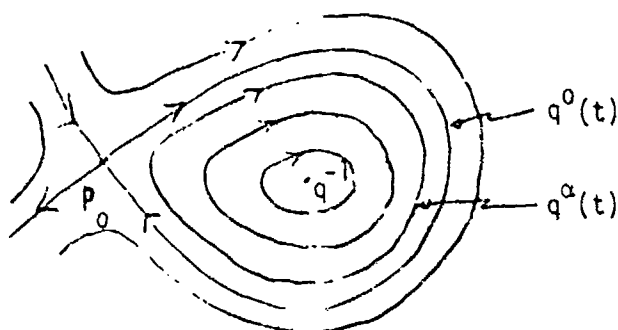


Figure 1.1. The unperturbed phase portrait of (1.3).

saddle points. Such orbits and cycles filled with periodic motions occur naturally in the phase planes of single degree of freedom nonlinear Hamiltonian systems; the pendulum:

$$\ddot{\theta} + \sin \theta = 0, \quad (1.7)$$

and Duffing's equation with negative linear stiffness:

$$\ddot{y} - y + y^3 = 0, \quad (1.8)$$

provide examples. We select Duffing's equation as our major example in subsequent sections because of its importance in mechanics, which we now

briefly indicate. For more details see Tseng and Dugundji [1971], Moon and Holmes [1979], Holmes [1979], Moon [1980]) and Holmes and Marsden [1981a].

Consider a slender elastic rod, clamped at one end in a rigid framework and constrained to move in a plane (Figure 1.2). Two magnets attached to the frame as indicated cause the beam to buckle either to the left or

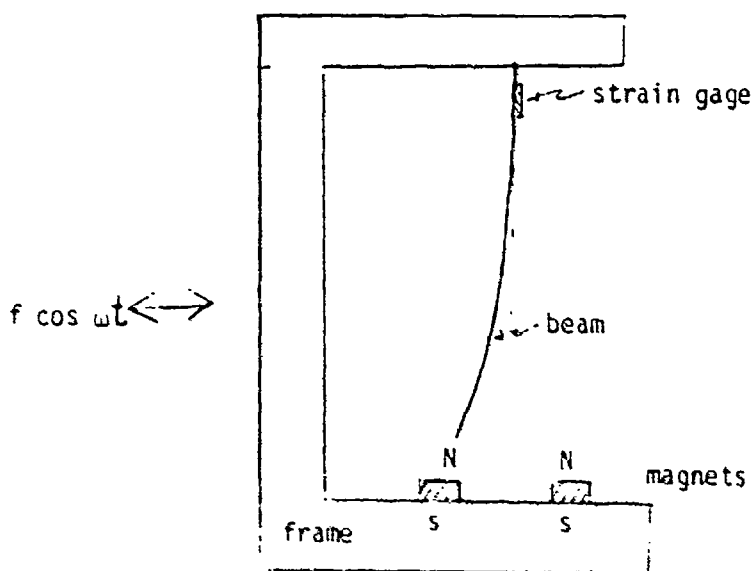


Figure 1.2. The magnetoelastic beam  
(from Moon and Holmes [1979])

right, the central position being unstable. The whole framework is now moved sinusoidally, so that the beam vibrates under 'inertial' excitation. For small excitation amplitudes, periodic motions are observed about either stable equilibrium, but as the amplitude is increased an apparently sudden transition to a 'chaotic snap through' motion is observed. In this state, the beam oscillates irregularly about first one and then the other equilibrium. An example of such a motion, measured from a strain gauge at the

beam's root, is shown in Figure 1.3.

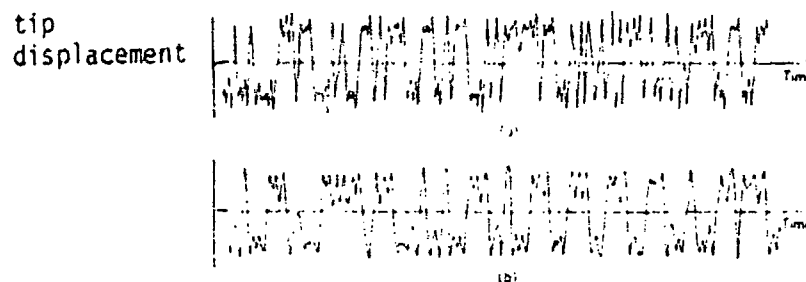


Figure 1.3. Nonperiodic motion of (a) the beam (from Moon and Holmes [1979]) and (b) the forced Duffing equation (from Holmes [1979]).

As shown by Moon and Holmes [1979], the nonlinear partial differential equations of an inextensible elastic rod in a nonuniform magnetic field can be truncated in a single mode Galerkin approximation to yield Duffing's equation:

$$\ddot{y} - y + y^3 = 0, \quad (1.9)$$

where  $y = y(t)$  is the (nondimensional) amplitude of the first mode of vibration and dissipation is ignored. On the introduction of weak dissipation (aerodynamic damping) and periodic forcing one obtains

$$\ddot{y} - y + y^3 = \epsilon(\gamma \cos \omega t - \delta \dot{y}), \quad (1.10)$$

which is the equation to be studied in Sections 4-6 of this article.

In the following section (2) we introduce some basic concepts from the qualitative theory of dynamical systems and then in Section 3 state and sketch the proofs of the results on perturbed integrable systems. In Section 4 we apply these results to Duffing's equation and then, in Section 5,

review more of the relevant abstract theory. Finally, in Section 6, we bring the abstract theory back to bear on Duffing's equation in an attempt to understand irregular motions such as that of Figure 1.3. In this article, although we assume some familiarity with the qualitative theory of dynamical systems and bifurcation theory, we do attempt to introduce or review most of the important aspects necessary for our study. Background material on differential dynamics may be found in Chillingworth [1976], and on bifurcation theory and nonlinear analysis in Iooss and Joseph [1980] and Chow and Hale [1981]. The latter reference, and Chow, Hale and Mallet-Paret [1980] contain many of the results of our Section 3, but presented from a different viewpoint. The book of Andronov, Vitt and Khaiken [1966] continues to provide the best background in nonlinear oscillations.

---

## 2. POINCARÉ MAPS AND INVARIANT MANIFOLDS

We consider the periodically forced system

$$\ddot{x} = f(x) + \varepsilon g(x, t); \quad x \in \mathbb{R}^2, \quad g(x, t) = g(x, t + T), \quad (2.1)$$

which can be rewritten as an autonomous system defined on the 'toroidal' phase space  $\mathbb{R}^2 \times S^1$ :

$$\left. \begin{aligned} \dot{x} &= f(x) + \varepsilon g(x, \theta) \\ \dot{\theta} &= 1 \end{aligned} \right\}; \quad (x, \theta) \in \mathbb{R}^2 \times S^1, \quad (2.2)$$

where  $S^1 = \mathbb{R}/T$ , the circle of length  $T$ . This naturally copes with the  $T$ -periodicity of the vector field. We next define a cross section for the flow:

$$\Sigma_{t_0} = \{(x, \theta) \in \mathbb{R}^2 \times S^1 \mid \theta = t_0 \in [0, T)\}, \quad (2.3)$$

and a first return or Poincaré map

$$P_{\varepsilon}^{t_0} : \Sigma_{t_0} \rightarrow \Sigma_{t_0} \quad (2.4)$$

obtained by following orbits  $(x(t + t_0), \theta(t + t_0))$  based at points  $(x(t_0), t_0) \in \Sigma_{t_0}$  to their next intersection with  $\Sigma_{t_0}$ . We have

$$P_{\varepsilon}^{t_0}(x(0)) = \pi \cdot (x(T + t_0), T + t_0) \quad (2.5)$$

where  $(x(t + t_0), t + t_0)$  is the solution of (2.2) based at  $(x(t_0), t_0)$  and  $\pi$  denotes projection onto the first factor. When  $\varepsilon = 0$ , the unperturbed Poincaré map  $P_0 = P_0^{t_0}$  is identical on every section  $\Sigma_{t_0}$  in view of the invariance of  $\dot{x} = f(x)$  under time translations.  $P_0$  is given simply by the time  $T$  flow map of the unperturbed equation

$$P_0(x(0)) = x^0(T), \quad (2.6)$$

where  $x^0(t)$  is the unperturbed solution of 2.1 based at  $x(0)$ . The perturbed maps  $P_\epsilon^{t_0}$  differ on different sections but any two are diffeomorphic and  $P_\epsilon^{t_0+nT} = P_\epsilon^{t_0}$ .

The Poincaré map captures the dynamics of (2.1) in the sense that  $T$ -periodic orbits of (2.1) correspond to fixed points of  $P_\epsilon^{t_0}$ ,  $mT$ -periodic subharmonics correspond to periodic cycles of period  $m$  and stability types correspond. The study of fixed and periodic points of the two dimensional map  $P_\epsilon^{t_0}$  is generally easier than that of the corresponding periodic motions in the three dimensional flow of (2.2).

Suppose that the map  $P$  (the sub- and superscripts will be dropped where they are not explicitly needed) has a hyperbolic fixed point  $p$ . This implies that the linearized map  $DP(p)$  has eigenvalues  $\lambda_i$  with modulus  $|\lambda_i| \neq 1$ ,  $i = 1, 2$ : a sink, a saddle or a source. The stable manifold theorem (cf. Chillingworth [1976], Hartman [1973], Hirsh-Pugh-Shub [1977]) and Appendix A) asserts that, in a neighborhood  $U(p)$  there are smooth local stable manifolds  $W_{loc}^s(p)$ ,  $W_{loc}^u(p)$  tangent to the eigenspaces  $E^s$ ,  $E^u$  of the linearized problem

$$\dot{\xi} = DP(p)\xi, \quad (2.7)$$

and of dimension  $s, u$  ( $s + u = 2$  here) where  $s$  (resp.  $u$ ) is the number of eigenvalues with moduli less than 1 (resp. greater than 1). These manifolds are simply defined as the sets of points asymptotic to  $p$  under forward (resp. backward) iterations of  $P$

$$W_{loc}^s(p) = \{x \in U(p) | P^n(x) \in U(p), \forall n, \text{ and } P^n(x) \rightarrow p \text{ as } n \rightarrow \infty\},$$

$$W_{loc}^u(p) = \{x \in U(p) | P^{-n}(x) \in U(p), \forall n, \text{ and } P^{-n}(x) \rightarrow p \text{ as } n \rightarrow \infty\},$$

and global manifolds are obtained by iteration of the local manifolds backwards (resp. forwards):

$$W^S(p) = \bigcup_{n \geq 0} P^{-n}(W_{loc}^S(p)) ,$$

$$W^U(p) = \bigcup_{n \geq 0} P^n(W_{loc}^U(p)) .$$

Of course if both eigenvalues lie inside (resp. outside) the unit circle then  $W^U(p)$  (resp.  $W^S(p)$ ) is empty.

We note that while locally they are smooth submanifolds of  $\mathbb{R}^2$ ,  $W^S(p)$  and  $W^U(p)$  are not necessarily submanifolds globally, since they may wind back and forth and accumulate on themselves, as they do in the specific examples to follow (Figure 2.1).

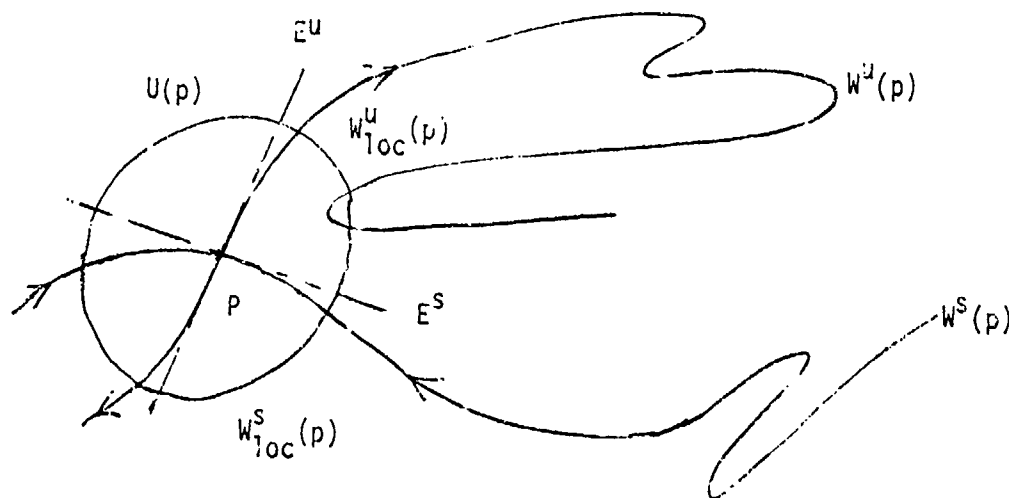


Figure 2.1 Stable and unstable manifolds

It is important to realize that the orbit of a map  $\{P^n(x)\}$  is a sequence of points, rather than a curve as in the case of orbits of flows arising from vector fields. Each of the curves  $W^S(p)$ ,  $W^U(p)$  in Figure 2.1 contains a one parameter family of such orbits. Uniqueness of solutions



implies that the stable (or unstable) manifolds  $W^s(p_1), W^s(p_2)$  of two distinct points cannot intersect, and nor can such a manifold intersect itself.

A homoclinic point  $q$  to a hyperbolic saddle point  $p$  is a point whose orbit approaches  $p$  under both forward and backward iterates of  $P$ :

$$\lim_{n \rightarrow \infty} P^n(q) = \lim_{n \rightarrow \infty} P^{-n}(q) = p,$$

thus  $q \in W^u(p) \cap W^s(p)$ . When two plane curves intersect at a point they generally do so transversely, so that their tangent vectors span  $\mathbb{R}^2$ . Such a point  $q \in W^u(p) \cap W^s(p)$  is called a transversal homoclinic point and will play an important rôle in our analysis. The existence of one such point immediately implies the existence of infinitely many, lying on the orbit  $\{P^n(q)\}$ . Moreover, the  $\lambda$ -lemma (Palis [1969], Newhouse [1980] and Appendix A) implies that  $W^u(p)$  accumulates upon itself, and  $W^s(p)$  accumulates upon itself, leading to the complicated structure of Figure 2.2.

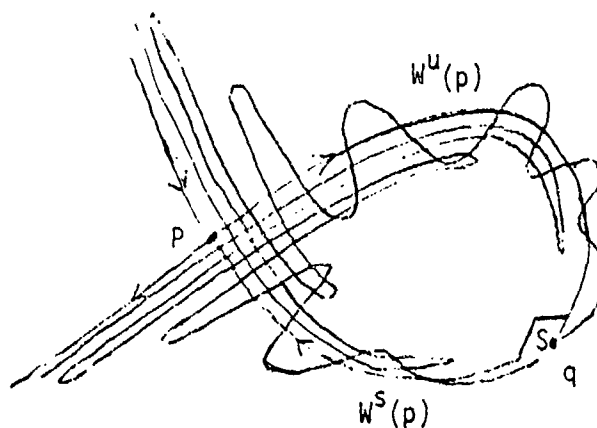


Figure 2.2. Transverse homoclinic orbits. See Section 5.1 for discussion of the rôle of the rectangle  $S$ .

In Section 5 we discuss some implications of such homoclinic orbits, but first we develop a method for their detection in specific examples.

### 3. PERTURBATIONS OF INTEGRABLE SYSTEMS: MELNIKOV'S METHOD

The basic ideas to be developed in this section are due to Melnikov [1963]. More recently Chow, Hale and Mallet-Paret [1980] have obtained similar results using alternative methods, and Holmes and Marsden [1981a,b,c] have applied the method to certain infinite dimensional flows arising from partial differential equations and to multidegree of freedom autonomous Hamiltonian systems. The basic idea is to make use of the globally computable solutions of the unperturbed integrable system in the computation of perturbed solutions. To do this we must first ensure that the perturbation calculations are uniformly valid on arbitrarily long or semi-infinite time intervals.

First we make our assumptions precise. We consider systems of the form

$$\dot{x} = f(x) + \epsilon g(x, t); \quad x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2, \quad (3.1)$$

where  $f = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ ,  $g = \begin{pmatrix} g_1(x, t) \\ g_2(x, t) \end{pmatrix}$  are sufficiently smooth ( $C^r$ ,  $r \geq 2$ ) and bounded on bounded sets and  $g$  is  $T$ -periodic in  $t$ . For simplicity we assume that the unperturbed system is Hamiltonian with  $f_1 = \frac{\partial H}{\partial v}$ ,  $f_2 = -\frac{\partial H}{\partial u}$ . The non-Hamiltonian case is considered by Melnikov [1963] and Holmes [1980]. Specific assumptions on the unperturbed flow are:

- A1. For  $\epsilon = 0$  (3.1) possesses a homoclinic orbit  $q^0(t)$  to a hyperbolic saddle point  $p_0$ .
- A2. Let  $\Gamma^0 = \{q^0(t) | t \in \mathbb{R}\} \cup \{p_0\}$ . The interior of  $\Gamma^0$  is filled with a continuous family of periodic orbits  $q^\alpha(t)$ ,  $\alpha \in (-1, 0)$ . Letting  $d(x, \Gamma) = \inf_{q \in \Gamma} |x - q|$  we have  $\lim_{\alpha \rightarrow 0} \sup_{t \in \mathbb{R}} d(q^\alpha(t), \Gamma) = 0$ .

A3. Let  $h_\alpha = H(q^\alpha(t))$  and  $T_\alpha$  be the period of  $q^\alpha(t)$ . Then  $T_\alpha$  is a differentiable function of  $h_\alpha$  and  $dT_\alpha/dh_\alpha > 0$  inside  $\Gamma^0$ .

We note that A2 and A3 imply that  $T_\alpha \xrightarrow{\text{monotonically}} \infty$  as  $\alpha \rightarrow 0$ . Many of the results to follow can be proved under less restrictive assumptions. In what follows we indicate the proofs but omit some details. See Greenspan [1981] for more information.

### 3.1 Bifurcations to Homoclinic Orbits

Lemma 3.1. Under the above assumptions, for  $\varepsilon$  sufficiently small (3.1) has a unique hyperbolic periodic orbit  $\gamma_\varepsilon^0(t) = p_0 + O(\varepsilon)$ . Correspondingly, the Poincaré map  $p_\varepsilon^0$  has a unique hyperbolic saddle point  $p_\varepsilon^0 = p_0 + O(\varepsilon)$ .

Proof. This is a straightforward application of the implicit function theorem, our assumptions implying that  $DP_0(p_0)$  does not contain 1 in its spectrum and hence that  $\text{Id} - DP_0(p_0)$  is invertible and there is a smooth curve of fixed points in  $(x, \varepsilon)$  space passing through  $(p_0, 0)$ . ■

Lemma 3.2. The local stable and unstable manifolds  $W_{\text{loc}}^s(\gamma_\varepsilon)$ ,  $W_{\text{loc}}^u(\gamma_\varepsilon)$  of the perturbed periodic orbit are  $C^r$  close to those of the unperturbed periodic orbit  $p_0 \times S^1$ . Moreover, orbits  $q_\varepsilon^s(t, t_0)$ ,  $q_\varepsilon^u(t, t_0)$  lying in  $W^s(\gamma_\varepsilon)$ ,  $W^u(\gamma_\varepsilon)$  and based on  $\Sigma^{t_0}$  can be expressed as follows, with uniform validity in the indicated time intervals.

$$\begin{aligned} q_\varepsilon^s(t, t_0) &= q^0(t - t_0) + \varepsilon q_1^s(t, t_0) + O(\varepsilon^2), \quad t \in [t_0, \infty); \\ q_\varepsilon^u(t, t_0) &= q^0(t - t_0) + \varepsilon q_1^u(t, t_0) + O(\varepsilon^2), \quad t \in (-\infty, t_0]. \end{aligned} \quad (3.2)$$

Proof. The existence of the perturbed manifolds follows from the invariant manifold theory of Hirsch, Pugh and Shub [1977], cf. Hartman [1973]. We fix a  $\nu$ -neighborhood ( $0 \leq \varepsilon \ll \nu \ll 1$ )  $U_\nu$  of  $p_0$  inside which the local perturbed manifolds and their tangent spaces are  $\varepsilon$ -close to those of the unperturbed flow (or Poincaré map). A standard Gronwall estimate shows that perturbed orbits starting within  $O(\varepsilon)$  of  $q^0(0)$  remain within  $O(\varepsilon)$  of  $q^0(t - t_0)$  for finite times and hence that one can follow any such orbit from an arbitrary point  $q^0(0)$  on  $\Gamma^0$  outside  $U_\nu$  to the boundary of  $U_\nu$ , at, say  $t = t_1$ . Once in  $U_\nu$ , if the perturbed orbit  $q_\varepsilon^S$  is selected to lie in  $W^S(\gamma_\varepsilon)$ , then its behavior is governed by the exponential contraction associated with the linearized system. Straightforward estimates then show that  $|q_\varepsilon^S(t, t_0) - q^0(t - t_0)| = O(\varepsilon)$  for  $t \in (t_1, \infty)$ . Reversing time, one obtains a similar result for  $q_\varepsilon^U(t, t_0)$ . ■

Sanders [1980] was the first to work out the asymptotics in detail. This lemma implies that solutions lying in the stable manifold are uniformly approximated by the solution  $q_1^S$  of the first variational equation:

$$\dot{q}_1^S(t, t_0) = Df(q^0(t - t_0))q_1^S(t, t_0) + g(q^0(t - t_0), t). \quad (3.3)$$

A similar expression holds for  $q_1^U(t, t_0)$ . Note that the initial time,  $t_0$  appears explicitly since solutions of the perturbed systems are not invariant under arbitrary translations in time.

We next define the separation of the manifolds  $W^U(p_\varepsilon^{t_0}), W^S(p_\varepsilon^{t_0})$  on the section  $\Sigma^{t_0}$  at the point  $q^0(0)$  as

$$d(t_0) = q_\varepsilon^U(t_0) - q_\varepsilon^S(t_0), \quad (3.4)$$

where  $q_\varepsilon^U(t_0) \stackrel{\text{def}}{=} q_\varepsilon^U(t_0, t_0), q_\varepsilon^S(t_0) \stackrel{\text{def}}{=} q_\varepsilon^S(t_0, t_0)$  are the unique points

on  $W^u(p_\epsilon^0)$ ,  $W^s(p_\epsilon^0)$  'closest' to  $p_\epsilon^0$  and lying on the normal  $f^\perp(q^0(0)) = (f_2(q^0(0)), f_1(q^0(0)))^T$  to  $\Gamma$  at  $q^0(0)$ . The  $C^r$  closeness of the manifolds to  $\Gamma$ , and Lemma 2 then imply that

$$d(t_0) = \epsilon \frac{f(q_0) \wedge (q_1^u(t_0) - q_1^s(t_0))}{|f(q_0(0))|} + O(\epsilon^2) \quad (3.5)$$

Here the wedge product is defined  $a \wedge b = a_1 b_2 - a_2 b_1$  and  $f \wedge (q_1^u - q_1^s)$  is the projection of  $q_1^u - q_1^s$  onto  $f^\perp$ , cf. Figure 3.1.

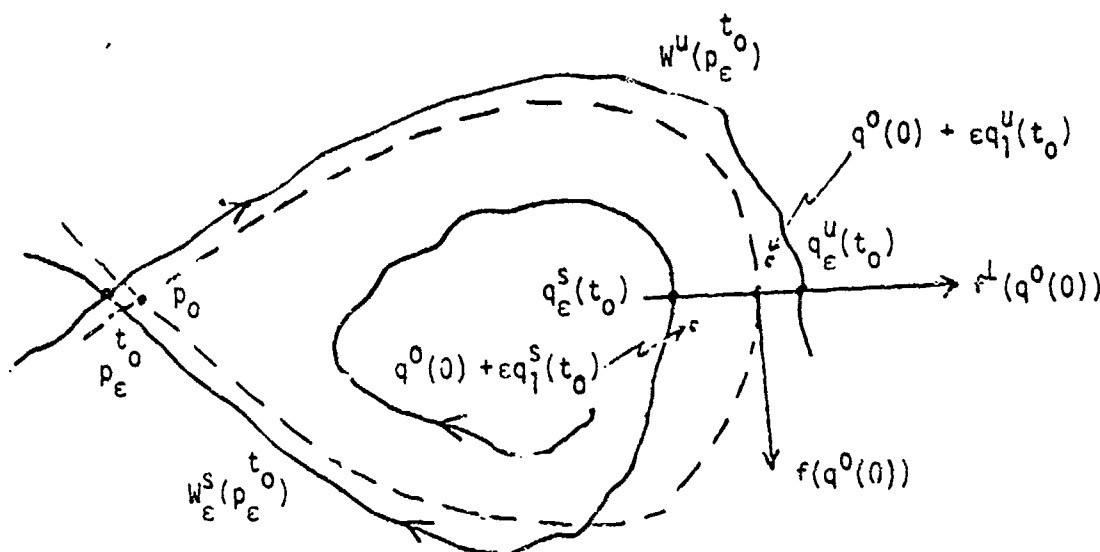


Figure 3.1 The perturbed manifolds and the distance function.

Finally we define the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t) dt. \quad (3.6)$$

Theorem 3.3. If  $M(t_0)$  has simple zeros and maxima and minima of  $O(1)$ , then for  $\epsilon > 0$  sufficiently small  $W^u(p_\epsilon^0)$  and  $W^s(p_\epsilon^0)$  intersect transversely.  
If  $M(t_0)$  remains bounded away from zero then  $W^u(p_\epsilon^0) \cap W^s(p_\epsilon^0) = \emptyset$ .

Proof. Consider the time dependent distance function

$$\Delta(t, t_0) = f(q^0(t - t_0)) \wedge (q_1^u(t, t_0) - q_1^s(t, t_0)) \quad (3.7)$$

$$\stackrel{\text{def}}{=} \Delta^u(t, t_0) - \Delta^s(t, t_0)$$

and note that  $d(t_0) = \varepsilon \Delta(t_0, t_0) / |f(q^0(u))| + O(\varepsilon^2)$ , from (3.5). We compute

$$\begin{aligned} \frac{d}{dt} \Delta^s(t, t_0) &= Df(q^0(t - t_0)) q_0(t - t_0) \wedge q_1^s(t, t_0) \\ &\quad + f(q^0(t - t_0)) \wedge \dot{q}_1^s(t, t_0). \end{aligned}$$

Using (3.3) and the fact that  $\dot{q}_0 = f(q_0)$  this yields

$$\begin{aligned} \Delta^s &= Df(q^0) f(q^0) \wedge q_1^s + f(q^0) \wedge (Df(q^0) q_1^s + g(q^0, t)) \\ &= \text{trace } Df(q^0) \Delta^s + f(q^0) \wedge g(q^0, t) \end{aligned} \quad (3.8)$$

But, since  $f$  is Hamiltonian,  $\text{trace } Df \equiv 0$ . Integrating (3.8) from  $t_0$  to  $\infty$  we have

$$\Delta^s(\infty, t_0) - \Delta^s(t_0, t_0) = \int_{t_0}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t) dt. \quad (3.9)$$

However  $\Delta^s(\infty, t_0) = \lim_{t \rightarrow \infty} f(q^0(t - t_0)) \wedge q_1^s(\infty, t_0)$  and  $\lim_{t \rightarrow \infty} q^0(t - t_0) = p_0$ ,

so that  $\lim_{t \rightarrow \infty} f(q^0(t - t_0)) = 0$  while  $q_1^s(\infty, t_0)$  is bounded, from Lemma

3.2. thus  $\Delta^s(\infty, t_0) = 0$  and (3.9) gives us a formula for  $\Delta^s(t_0, t_0)$ . A similar calculation gives

$$\Delta^u(t_0, t_0) = \int_{-\infty}^{t_0} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t) dt, \quad (3.10)$$

and addition of (3.9) and (3.10) and use of (3.5) yields

$$d(t_0) = \epsilon M(t_0) / |f(q^0(0))| + O(\epsilon^2). \quad (3.11)$$

Since  $|f(q^0(0))| = O(1)$ ,  $M(t_0)$  provides a good measure of the separation of the manifolds at  $q^0(0)$  on  $\Sigma^{t_0}$ . In particular, if  $M(t_0)$  oscillates about zero with maxima and minima independent of  $\epsilon$ , then  $q_\epsilon^u(t_0)$  and  $q_\epsilon^s(t_0)$  must change their orientation with respect to  $f^\perp(q^0(0))$  as  $t_0$  varies and hence  $W_\epsilon^u(P_\epsilon^{t_0})$  and  $W_\epsilon^s(P_\epsilon^{t_0})$  intersect. If the zeros are simple  $\left(\frac{dM}{dt_0} \neq 0\right)$ , then it follows that the intersections are transversal. Conversely, if no zeros exist, then  $q_\epsilon^u$  and  $q_\epsilon^s$  retain the same orientation and hence the manifolds do not intersect. ■

Remark 1. We note that  $M(t_0)$  is  $T$ -periodic in  $t_0$ , as it should be,

since the maps  $P_\epsilon^{t_0}$  and  $P_\epsilon^{t_0+T}$  are identical and thus  $d(t_0) = d(t_0 + T)$ . In computing  $M(t_0)$  we are effectively standing at a fixed point  $q^0(0)$  on a moving cross section  $\Sigma^{t_0}$  and watching the perturbed manifolds oscillate as  $t_0$  varies. In his analysis, Greenspan [1981] fixes the section and moves the base point  $q^0(0)$ .

2. If the perturbation  $g$  is derived from a (time dependent) Hamiltonian function  $G$ ,  $g_1 = \frac{\partial G}{\partial v}$ ,  $g_2 = -\frac{\partial G}{\partial u}$ , then we have

$$M(t_0) = - \int_{-\infty}^{\infty} (H(q^0(t - t_0)), G(q^0(t - t_0), t)) dt, \quad (3.12)$$

where  $(\cdot, \cdot)$  denotes the Poisson bracket. This is a natural formula if one recalls that the first variation of the unperturbed Hamiltonian energy



$H$  will be obtained by integrating its evolution equation

$$\dot{H} = \{H, G\} \quad (3.13)$$

along the unperturbed orbit  $q^0(t - t_0)$ , cf. Arnold [1964].

3. If  $g = g(x)$  is not explicitly time dependent then we have, using Green's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0)) dt &= \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1) dt \\ &= \int_{\Gamma} (g_2(u^0, v^0) \dot{u}^0 - g_1(u^0, v^0) \dot{v}^0) dt \\ &= \int_{\Gamma} g_2(u, v) du - g_1(u, v) dv \\ &= \int_{\text{int } \Gamma} \text{trace } Dg(x) dx. \end{aligned} \quad (3.14)$$

Thus the formula obtained in Andronov et al [1971] is a special (planar) case of the more general Melnikov function which describes the 'splitting' of the perturbed saddle separatrices.

We now turn to the case in which the perturbation  $g = g(x, t; \mu)$  depends upon parameters  $\mu \in \mathbb{R}^p$ . For simplicity we take  $p = 1$ .

Corollary 3.4. Consider the parametrized family  $\dot{x} = f(x) + \epsilon g(x, t; \mu)$ ,  $\mu \in \mathbb{R}$  and let hypotheses A1-3 hold. Suppose that the Melnikov function  $M(t_0, \mu)$  has a quadratic zero  $M(\tau, \mu_B) = \frac{\partial M}{\partial t_0}(\tau, \mu_B) = 0$  but  $\frac{\partial^2 M}{\partial t_0^2}(\tau, \mu_B) \neq 0$  and  $\frac{\partial M}{\partial \mu}(\tau, \mu_B) \neq 0$ . Then  $\mu_B = \mu_0 + O(\epsilon)$  is a bifurcation value for which quadratic homoclinic tangencies occur in the family of systems.

Proof. By hypothesis, using (3.5), we have

$$d(t_0, \mu) = \epsilon \{ \alpha(\mu - \mu_b) + \beta(t_0 - \tau)^2 \} + O(\epsilon |\mu - \mu_b|^2) + O(\epsilon^2) \quad (3.15)$$

where we have expanded in a Taylor series about  $(t_0, \mu) = (\tau, \mu_b)$  and  $\alpha, \beta$  are finite constants. Taking  $\epsilon$  sufficiently small we find that  $d(t_0, \mu)$  has a quadratic zero with respect to  $t_0$  for some  $\mu_b$  near  $\mu_b$ , and hence that  $W^u(p_\epsilon^\tau), W^s(p_\epsilon^\tau)$  have a quadratic tangency near  $q^0(0)$  on  $\Sigma^\tau$ . ■

This result is important, since it permits us to verify in specific examples one of the hypotheses of Newhouse's [1979] theorem on wild hyperbolic sets (see Section 5).

We next turn to the perturbations of the periodic motions  $q^\alpha$  inside  $\Gamma$ .

### 3.2 Bifurcations of Subharmonics

Once more we start with a perturbation lemma:

Lemma 3.5. Let  $q^j(t - t_0)$  be a periodic orbit of the unperturbed system based on  $\Sigma^{t_0}$  with period  $T_a$ . Then there exists a perturbed orbit  $q_\epsilon^j(t, t_0)$ , not necessarily periodic, which can be expressed as

$$q_\epsilon^j(t, t_0) = q^j(t - t_0) + \epsilon q_1^j(t, t_0) + O(\epsilon^2) \quad (3.16)$$

uniformly in  $t \in [t_0, t_0 + T_a]$ , for  $\epsilon$  sufficiently small and all  $a \in (-1, 0)$ .

Proof. The proof relies heavily upon the geometrical structure of the perturbed stable and unstable manifolds established in Lemma 3.2. We again

fix a neighborhood  $U_V$  and take a curve of initial conditions  $q^\alpha(0) \in \Sigma^{t_0}$  not lying in  $U_V$  and with  $\lim_{\alpha \rightarrow 0} q^\alpha(0) = q^0(0)$ . Any orbit  $q^\alpha(t - t_0)$  starting on such a curve takes a finite time to reach the boundary of  $U_V$  as  $t$  increases or decreases and hence we have  $|q_\epsilon^\alpha(t, t_0) - q^\alpha(t - t_0)| = O(\epsilon)$  for  $t \in (t_0 - t_1, t_0 + t_2)$ , say. Once in  $U_V$  the perturbed and unperturbed orbits may take arbitrarily long to pass through and exit, since, as  $\alpha \rightarrow 0$ , they pass arbitrarily close to the saddle point  $p_0$  (or to  $\gamma_\epsilon$ ). However, for fixed  $\epsilon = \epsilon_0$  we can find a set of orbits lying sufficiently close to the stable and unstable manifolds which remain within  $O(\epsilon)$  of those manifolds until they enter a  $c\epsilon$  neighborhood  $U_{c\epsilon}$  of  $p$ , which is chosen to contain  $\gamma_\epsilon$ , Figure 3.2. This implies that we can extend our estimate uniformly to  $t \in (t_0 - t_3, t_0 + t_4)$ , where  $t_3 + t_4$  is the time required for the unperturbed orbit  $q^\alpha$  to pass from the boundary of  $U_{c\epsilon}$  and return to it. It remains to check that  $q_\epsilon^\alpha(t, t_0)$  and  $q^\alpha(t - t_0)$  remain within  $O(\epsilon)$  during the arbitrarily long passage

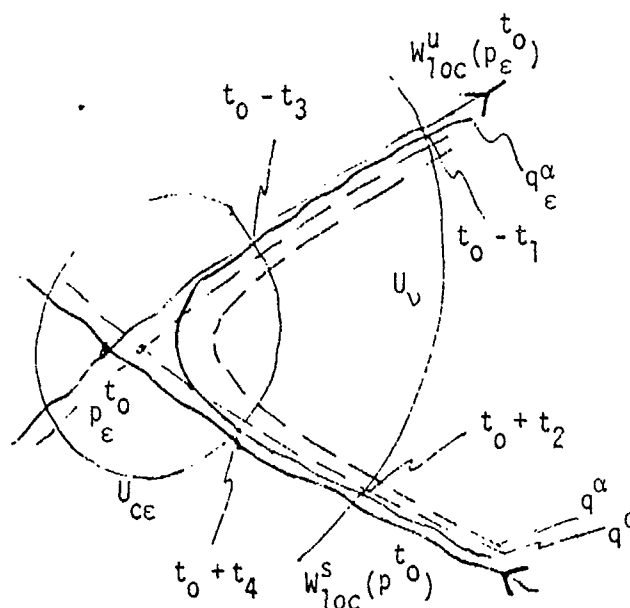


Figure 3.2. Orbits in  $U_V$  and  $U_{c\epsilon}$

through  $U_{c\epsilon}$ . This follows from the 'shear' on the flow near  $p$ ; one has to check that  $q_\epsilon^\alpha$  exits from  $U_{c\epsilon}$  'at the correct time' and since orbits passing near  $\gamma_\epsilon$  take arbitrarily long to exit, and those near the boundary of  $U_{c\epsilon}$  arbitrarily short times, at least one orbit can be found for any given (unperturbed) time of passage. It follows that an initial condition  $q_\epsilon^\alpha(t_0) \in \Sigma^{t_0}$  can be picked with  $O(\epsilon)$  of  $q^\alpha(0)$  and  $q^0(0)$  such that the orbit  $q_\epsilon^\alpha(t_0)$  will remain within  $O(\epsilon)$  of  $q^\alpha(t - t_0)$  (and  $q_\epsilon^s(t - t_0)$ ) until it reaches  $U_{c\epsilon}$ . It will then 'transfer its allegiance' to  $q_\epsilon^u(t - t_0)$  until it once more reaches an  $\epsilon$ -neighborhood of  $q^\alpha(0)$ . Throughout it remains within  $O(\epsilon)$  of  $q^\alpha(t - t_0)$ . This takes care of orbits with unperturbed periods  $T_\alpha$  larger than some  $T'_\alpha = T'_\alpha(\epsilon_0)$  depending on  $\epsilon_0$ . For orbits with periods shorter than  $T'_\alpha(\epsilon_0)$  a standard Gronwall estimate ensures  $\epsilon$ -closeness. Then we have our result for all  $T_\alpha$  and  $\epsilon = \epsilon_0$ . But since  $f$  and  $g$  are  $C^r$  the solution will vary smoothly in  $\epsilon$  and thus the result holds for all  $0 < \epsilon \leq \epsilon_0$ . For details see Greenspan [1981]. ■

We next define the subharmonic Melnikov function. Letting  $q^\alpha(t - t_0)$  be a periodic orbit of period  $\frac{mT}{n}$ , with  $m$  and  $n$  relatively prime, we set

$$M^{m/n}(t_0) = \int_0^{mT} f(q^\alpha(t - t_0)) \wedge g(q^\alpha(t - t_0), t) dt. \quad (3.17)$$

Theorem 3.6. If  $M^{m/n}(t_0)$  has simple zeros and maxima and minima of  $O(1)$ , and  $dT_\alpha/dh_\alpha \neq 0$ , then for  $0 < \epsilon \leq \epsilon(n)$  (3.1) has a subharmonic orbit of period  $mT$ . If  $n = 1$  then the result is uniformly valid in  $0 < \epsilon \leq \epsilon_0$ .

Proof. A calculation similar to that of Theorem 3.3 shows that

$$f(q^\alpha(0)) \wedge (q_\epsilon^\alpha(t_0 + mT, t_0) - q_\epsilon^\alpha(t_0, t_0)) = \epsilon \int_0^{mT} f(q^\alpha(t - t_0)) \wedge g(q^\alpha(t - t_0), t) dt / |f(q^\alpha(0))| + O(\epsilon^2). \quad (3.18)$$

Thus if  $M^{m/n}(t_0)$  has a simple zero then there is a perturbed orbit  $q_\epsilon^\alpha(t, t_0)$  which leaves  $q_\epsilon^\alpha(t_0)$  and returns to  $\Sigma^{t_0}$  at  $q_\epsilon^\alpha(t_0 + mT)$  with the vector  $q_\epsilon^\alpha(t_0 + mT) - q_\epsilon^\alpha(t_0) \subset \Sigma^{t_0}$  parallel to  $f(q^\alpha(0))$ ; i.e. the projection onto  $f^\perp(q^\alpha(0))$  vanishes. Letting  $M^\beta(t_0) = \int_0^{mT} f(q^\beta) \wedge g(q^\beta, t) dt$  for  $\beta$  near  $\alpha$ , it is clear that  $M^\beta$  depends smoothly on  $\beta$ . For  $\beta < \alpha, T_\beta < T_\alpha$  and for  $\beta > \alpha, T_\beta > T_\alpha$ ; it therefore follows that we can find perturbed orbits  $q_\epsilon^{\beta_1}, q_\epsilon^{\beta_2}, \beta_1 < \alpha < \beta_2$  such that the vectors  $q_\epsilon^{\beta_i}(t_0 + mT) - q_\epsilon^{\beta_i}(t_0) \subset \Sigma^{t_0}$  are parallel to  $f(q^{\beta_i}(0))$ , but that they have opposite orientations. Thus the curve of initial conditions connecting  $q_\epsilon^{\beta_1}(t_0)$  to  $q_\epsilon^{\beta_2}(t_0)$  is mapped back to the section  $\Sigma^{t_0}$  under  $m$  iterates of  $P_\epsilon^{t_0}$  as indicated in Figure 3.3. It follows that  $(P_\epsilon^{t_0})^m$  has a fixed point near  $q^\alpha(0)$ , and hence that there is a subharmonic of order  $m/n$ . The nonuniformity for  $n > 1$  arises because Lemma 3.5 applies only to orbits with times of length  $T_\alpha = mT$  making one pass through  $U_V(p)$  and 'ultrasubharmonics' of period  $mT/n$  make  $n$  passes through  $U_V(p)$ . ■

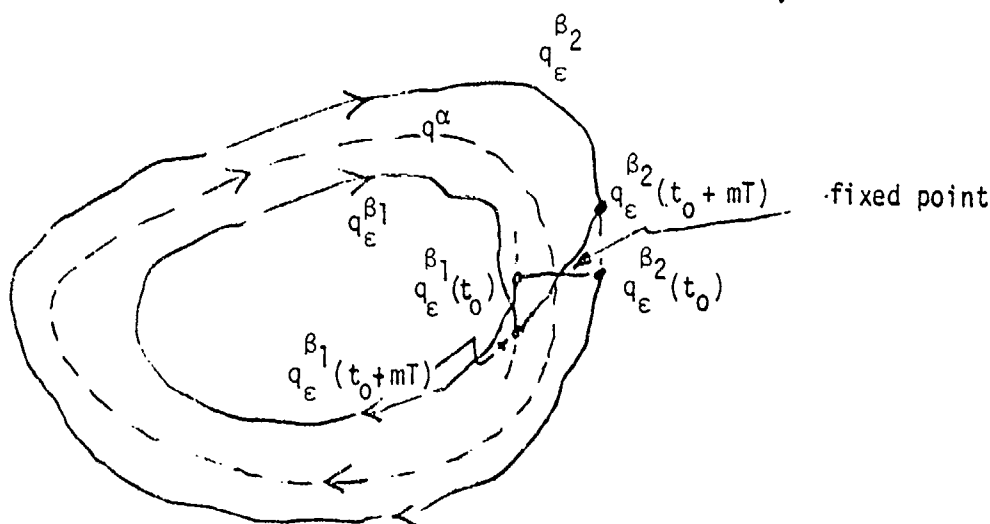


Figure 3.3. The proof of Theorem 3.6.

We also have a bifurcation result analogous to Corollary 3.4:

Corollary 3.7. Consider the parametrized family  $\dot{x} = f(x) + \epsilon g(x, t; \mu)$ ,  $\mu \in \mathbb{R}$ , and let hypotheses A1-3 hold. Suppose that  $M^{m/n}(t_0, \mu)$  has a quadratic zero  $M^{m/n} = \frac{\partial M^{m/n}}{\partial t_0} = 0$ ;  $\frac{\partial^2 M^{m/n}}{\partial t_0^2} \neq 0$ ,  $\frac{\partial M^{m/n}}{\partial \mu} \neq 0$  at  $\mu = \mu_b$ . Then  $\mu_{m/n} = \mu_b + O(\epsilon)$  is a bifurcation value at which saddle-nodes occur.

Proof. The proof is identical to that of Corollary 3.4. We will consider the stability types of the orbits created in this bifurcation in Section 3.3. ■

The next result is a generalization of one obtained by Chow, Hale and Mallet Paret [1980]. It implies that the homoclinic bifurcation is the limit of a countable sequence of subharmonic saddle-node bifurcations.

Theorem 3.8. Let  $M^{m/1}(t_0) = M^m(t_0)$ , then

$$\lim_{m \rightarrow \infty} M^m(t_0) = M(t_0). \quad (3.19)$$

Proof. We must show that the integral

$$M^m(t_0) = \int_{-\frac{mT}{2}}^{\frac{mT}{2}} f(q^\alpha(t - t_0)) \wedge g(q^\alpha(t - t_0), t) dt \quad (3.20)$$

converges to

$$M(t_0) = \int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t) dt, \quad (3.21)$$

as  $\alpha \rightarrow 0$  and  $m \rightarrow \infty$ . (Note that the periodicity of  $M^m(t_0)$  implies that we can change the limits from  $0 \rightarrow mT$  to  $-\frac{mT}{2} \rightarrow \frac{mT}{2}$ .) Letting

$\Gamma^\alpha = \{q^\alpha(t) | t \in [0, T_\alpha]\}$  and  $\Gamma^0 = \{q^0(t) | t \in \mathbb{R}\} \cup \{p_0\}$  we select a neighborhood  $U_\nu(p)$  such that the arc length of  $\Gamma^0$ ,  $\Gamma^\alpha \cap U_\nu(p)$  is less than  $\nu$ . Choose  $\tau$  such that  $q^0(-\tau)$  and  $q^0(\tau)$  both lie within  $U_\nu$ . Then for  $\alpha$  close enough to zero,  $q^\alpha(\pm\tau)$  also lie in  $U_\nu$ . We have

$$\begin{aligned} M(t_0) - M^m(t_0) &= \left[ \int_{-\tau}^{\tau} f(q^0) \wedge g(q^0, t) dt - \int_{-\tau}^{\tau} f(q^\alpha) \wedge g(q^\alpha, t) dt \right] \\ &+ \left[ \int_{-\infty}^{-\tau} f(q^0) \wedge g(q^0, t) dt + \int_{\tau}^{\infty} f(q^0) \wedge g(q^0, t) dt \right. \\ &\left. - \int_{-\frac{mT}{2}}^{-\tau} f(q^\alpha) \wedge g(q^\alpha, t) dt - \int_{\tau}^{\frac{mT}{2}} f(q^\alpha) \wedge g(q^\alpha, t) dt \right]. \end{aligned} \quad (3.22)$$

The smoothness of  $f$  and  $g$  and continuity of solutions with respect to initial conditions implies that, given  $\nu > 0$ , there is an  $\alpha' < 0$  such that, for  $\alpha \in (\alpha', 0)$  the first [bracketed] term of (3.22) is less than  $\nu$ . Clearly, as  $\alpha' \rightarrow 0^-$ ,  $m \rightarrow \infty$ . The second term may be expressed as the difference between two integrals over arcs of  $\Gamma^0$  and  $\Gamma^\alpha$ . Using the arc-length increment  $ds = \sqrt{\dot{u}^2 + \dot{v}^2} dt = |f(q^0)| dt$  on  $\Gamma^0$  and  $ds = |f(q^\alpha)| dt$  on  $\Gamma^\alpha$ , the second term becomes

$$\oint_{q^0(-\tau)}^{q^0(\tau)} f(q^0) \wedge g(q^0, t) \frac{ds}{|f(q^0)|} - \oint_{q^\alpha(-\tau)}^{q^\alpha(\tau)} f(q^\alpha) \wedge g(q^\alpha, t) \frac{ds}{|f(q^\alpha)|}. \quad (3.23)$$

Our assumptions on  $g$  imply that  $\sup_{\substack{q \in B_\nu(p) \\ t \in \mathbb{R}}} |g(q^\alpha, t)| = K < \infty$ , and thus that the second term is bounded by  $2K\nu$ . Hence, for  $\alpha \in (\alpha', 0)$

$$|M(t_0) - M^m(t_0)| < (2K + 1)\nu$$

and  $|M(t_0) - M^m(t_0)| \rightarrow 0$  as  $\alpha \rightarrow 0$ . ■

### 3.3 Higher Order Terms and Stability

We next develop a perturbation method which enables us to compute the global structure of the perturbed Poincaré map  $P_\epsilon^{t_0}$ , and to determine how the sets of subharmonics and homoclinic orbits are related. Our starting point is Melnikov [1963, §7], although we have somewhat modified his transformations.

Since the unperturbed system is Hamiltonian, a symplectic change of coordinates to action angle variables can be found in the interior of  $\Gamma^0$ :

$$I = I(u, v) \quad \theta = \theta(u, v) . \quad (3.24)$$

Under this change of coordinates (3.1) becomes

$$\dot{I} = \epsilon \left( \frac{\partial I}{\partial u} g_1 + \frac{\partial I}{\partial v} g_2 \right) \stackrel{\text{def}}{=} \mathcal{E}F(I, \theta, t) \quad (3.25)$$

$$\dot{\theta} = \Omega(I) + \epsilon \left( \frac{\partial \theta}{\partial u} g_1 + \frac{\partial \theta}{\partial v} g_2 \right) \stackrel{\text{def}}{=} G(I, \theta, t)$$

where  $\Omega(I^\alpha) = \frac{\partial H}{\partial I}(I^\alpha) = \frac{2\pi}{T_\alpha}$  is the angular frequency of the unperturbed orbit  $q^\alpha(t - t_0)$  with action  $I^\alpha = I(q^\alpha)$ . We now consider small perturbations of a resonant orbit  $T_\alpha = \frac{mT}{n}$ . Letting

$$I = I^\alpha + \sqrt{\epsilon}h$$

$$\theta = \Omega(I^\alpha)t + \phi = \left( \frac{2\pi n}{mT} \right) t + \phi \stackrel{\text{def}}{=} \Omega^\alpha t + \phi, \quad (3.26)$$

we obtain



$$\begin{aligned}\dot{h} &= \sqrt{\epsilon} F(I^\alpha, \Omega^\alpha t + \phi, t) + \epsilon F'(I^\alpha, \Omega^\alpha t + \phi, t) h + O(\epsilon^{3/2}), \\ \dot{\phi} &= \sqrt{\epsilon} \Omega'(I^\alpha) h + \epsilon (G(I^\alpha, \Omega^\alpha t + \phi, t) + \Omega''(I^\alpha) h^2)\end{aligned}\quad (3.27)$$

where ' denotes  $\frac{\partial}{\partial I}$ . Here we have expanded in Taylor series and used the fact that  $\Omega' \neq 0$ , since  $dT_\alpha/dh_\alpha \neq 0$ . Noting that

$$\frac{\partial I}{\partial u} = \frac{\partial I}{\partial H} \frac{\partial H}{\partial u} = -\frac{1}{\Omega(I)} f_2 \quad \text{and} \quad \frac{\partial I}{\partial v} = \frac{1}{\Omega(I)} f_1,$$

the leading term of (3.27) can be rewritten

$$\begin{aligned}\dot{h} &= \sqrt{\epsilon} \frac{1}{\Omega^\alpha} f(q^\alpha(t - \phi/\Omega^\alpha)) + g(q^\alpha(t - \phi/\Omega^\alpha), t) \\ \dot{\phi} &= \sqrt{\epsilon} \Omega'(I^\alpha) h.\end{aligned}\quad (3.28)$$

Provided that  $\Omega'(I^\alpha)$  is bounded, for  $\sqrt{\epsilon}$  sufficiently small the averaging theorem (cf. Hale [1963]) can be applied to (3.28) to yield

$$\dot{\bar{h}} = \sqrt{\epsilon} \frac{1}{\Omega^\alpha} \frac{1}{mT} \int_0^{mT} f(q^\alpha(t - \phi/\Omega^\alpha)) + g(q^\alpha(t - \phi/\Omega^\alpha), t) dt$$

or

$$\begin{aligned}\dot{\bar{h}} &= \sqrt{\epsilon} \frac{1}{2\pi n} M^{m/n} \left( \frac{\bar{\phi}}{\Omega^\alpha} \right), \\ \dot{\bar{\phi}} &= \sqrt{\epsilon} \Omega'(I^\alpha) \bar{h}.\end{aligned}\quad (3.29)$$

Under the averaging theorem, the hyperbolic or elliptic fixed points of (3.29) correspond to small periodic motions of (3.27) and hence to subharmonics of order  $m/n$  of (3.1). It is, of course, no coincidence that necessary and sufficient condition for the existence of such fixed points is that the Melnikov function  $M^{m/n}$  have simple zeros and that  $\Omega'(I^\alpha)$ ,  $\Omega(I^\alpha) \neq 0$  ( $dT_\alpha/dh_\alpha \neq 0$ ).

We note that (3.29) is a structurally unstable Hamiltonian system with

Hamiltonian

$$H = \sqrt{\epsilon} \left( \frac{\Omega' h^2}{2} - V(\bar{\phi}) \right), \quad (3.30)$$

where  $V(\phi) = \frac{1}{2\pi n} \int M^{m/n}(\bar{\phi}/\Omega^\alpha) d\phi$ , and thus to determine the stability and the global behavior of orbits of the unperturbed system near the resonant orbit  $q^\alpha$ , we must investigate the terms of  $O(\epsilon)$ . Letting  $f \sim g = \frac{1}{2\pi n} M^{m/n} \left( \frac{\phi}{\Omega^\alpha} + \tilde{F}(\phi, t) \right)$  where  $\tilde{F}$  has period  $T$  and zero mean, the averaging transformation is

$$h \rightarrow \bar{h} + \sqrt{\epsilon} \int \tilde{F}(\bar{\phi}, t) dt; \quad \phi \rightarrow \bar{\phi}, \quad (3.31)$$

where the antiderivative is defined up to a  $t$ -independent term, generally taken to be zero. Using (3.31), (3.27) becomes

$$\begin{aligned} \dot{\bar{h}} &= \sqrt{\epsilon} \frac{1}{2\pi n} M^{m/n}(\bar{\phi}/\Omega^\alpha) + \epsilon \left( (F'(I^\alpha, \Omega^\alpha t + \bar{\phi}, t) \bar{h} - \frac{\partial}{\partial \phi} \int \tilde{F} dt \Omega' \bar{h}), \right. \\ &\quad \left. + O(\epsilon^{3/2}) \right) \\ \dot{\bar{\phi}} &= \sqrt{\epsilon} \Omega' \bar{h} + \epsilon (\Omega'' \bar{h}^2 + G(I^\alpha, \Omega^\alpha t + \bar{\phi}, t) + \Omega' \int \tilde{F} dt). \end{aligned} \quad (3.32)$$

Since  $\tilde{F}$  has zero mean (it is simply a sum of Fourier components),  $\int \tilde{F}$  and  $\frac{\partial}{\partial \phi} \int \tilde{F}$  also have zero mean and on a second application of averaging to the  $O(\epsilon)$  terms of (3.32) we obtain (dropping the bars)

$$\begin{aligned} \dot{h} &= \sqrt{\epsilon} \frac{1}{2\pi n} M^{m/n}(\phi/\Omega^\alpha) + \epsilon \bar{F}'(\phi) h, \\ \dot{\phi} &= \sqrt{\epsilon} \Omega' h + \epsilon (\Omega'' h^2 + \bar{G}(\phi)), \end{aligned} \quad + O(\epsilon^{3/2}), \quad (3.33)$$

where  $\bar{F}'$ ,  $\bar{G}$  are the averages of  $F'$  and  $G$ . As Morosov [1973] notes, this second order averaging generally suffices to determine the stability of the fixed points and hence of the bifurcating subharmonics, at least for  $\Omega' < \infty$  and  $\epsilon$  sufficiently small. However, as we shall see in our application in sections 4 and 6, one can also obtain global information on the

Poincaré map by considering the time  $T$  flow maps of the averaged systems (3.33) in the neighborhood of each resonant and nonresonant periodic orbit. These results on the full Poincaré map  $P_\epsilon$  follow from application of the averaging theorem (Hale [1969]).

#### 4. GLOBAL BIFURCATIONS OF DUFFING'S EQUATION

In this section we apply the results of Section 3 to the Duffing equation with weak sinusoidal forcing and damping. Written as a system, we have

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= u - u^3 + \varepsilon(\gamma \cos \omega t - \delta v),\end{aligned}\tag{4.1}$$

where the force amplitude,  $\gamma$ , frequency  $\omega$ , and the damping  $\delta$  are variable parameters and  $\varepsilon$  is a small scaling parameter. For  $\varepsilon = 0$  the system has centers at  $(u, v) = (\pm 1, 0)$  and a hyperbolic saddle at  $(0, 0)$ .

The level set

$$(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4} = 0\tag{4.2}$$

is composed of two homoclinic orbits,  $\Gamma_+^0, \Gamma_-^0$  and the point  $p_0 = (0, 0)$ . The unperturbed homoclinic orbits are given by

$$\begin{aligned}q_+^0(t - t_0) &= (\sqrt{2} \operatorname{sech}(t - t_0), -\sqrt{2} \operatorname{sech}(t - t_0) \tanh(t - t_0)), \\ q_-^0 &= -q_+^0.\end{aligned}\tag{4.3}$$

Within each of the loops  $\Gamma_{\pm}^0$  there is a one parameter family of periodic orbits which may be written

$$\begin{aligned}q_+^k(t - t_0) &= \left( \sqrt{\frac{2}{2 - k^2}} \operatorname{dn} \left( \frac{t - t_0}{\sqrt{2 - k^2}}, k \right), \frac{-\sqrt{2}k^2}{2 - k^2} \operatorname{sn} \left( \frac{t - t_0}{\sqrt{2 - k^2}}, k \right) \operatorname{cn} \left( \frac{t - t_0}{\sqrt{2 - k^2}}, k \right) \right), \\ q_-^k(t - t_0) &= -q_+^k(t - t_0),\end{aligned}\tag{4.4}$$

where  $\operatorname{sn}$ ,  $\operatorname{cn}$  and  $\operatorname{dn}$  are the Jacobi elliptic functions and  $k$  is the elliptic modulus. As  $k \rightarrow 1$   $q_{\pm}^k \rightarrow q_{\pm}^0 \cup \{0, 0\}$  and as  $k \rightarrow 0$   $q_{\pm}^k \rightarrow (\pm 1, 0)$ . We have selected initial conditions at  $t = t_0$ :

$$q_{\pm}^0(0) = (\pm\sqrt{2}, 0), \quad q_{\pm}^k(0) = \left( \pm \sqrt{\frac{2}{2-k^2}}, 0 \right). \quad (4.5)$$

The orbits lying outside  $\Gamma_+^0 \cup \{(0,0)\} \cup \Gamma_-^0$  are given by

$$\hat{q}^k(t - t_0) = \left[ \frac{\sqrt{2k^2}}{\sqrt{2k^2 - 1}} \operatorname{cn} \left( \frac{t - t_0}{\sqrt{2k^2 - 1}}, k \right), -\frac{\sqrt{2}k}{2k^2 - 1} \operatorname{sn} \left( \frac{t - t_0}{\sqrt{2k^2 - 1}}, k \right) \right] \times \\ \operatorname{dn} \left( \frac{t - t_0}{\sqrt{2k^2 - 1}}, k \right), \quad (4.6)$$

where  $k \in (1/\sqrt{2}, 1)$  and  $\hat{q}^k \rightarrow q_+^0 \cup \{(0,0)\} \cup q_-^0$  as  $k \rightarrow 1$  and  $\hat{q}^k$  becomes unbounded as  $k \rightarrow 1/\sqrt{2}$ .

We note that the Hamiltonian (4.2) can be rewritten within  $\Gamma_+^0$  (or  $\Gamma_-^0$ ) in terms of the elliptic modulus  $k$ .

$$H(q^k) = \frac{k^2 - 1}{(2 - k^2)^2} \stackrel{\text{def}}{=} h_k. \quad (4.7)$$

Moreover, the period of these orbits is given by

$$T_k = 2K(k)\sqrt{2 - k^2}, \quad (4.8)$$

where  $K(k)$  is the complete elliptic integral of the first kind.  $T$  increases monotonically in  $k$  with  $\lim_{k \rightarrow 0} T_k = \sqrt{2\pi}$ ,  $\lim_{k \rightarrow 1} T_k = \infty$  and

$$\frac{dT_k}{dh_k} = \frac{dT_k/dk}{dH/dk} > 0 \quad (4.9a)$$

and

$$\lim_{k \rightarrow 1} \frac{dT_k}{dh_k} = \infty. \quad (4.9b)$$

We first compute the Melnikov function for  $q_+^0$  (the computation for  $q_-^0$  is identical):

$$\begin{aligned}
M(t_0, \gamma, \delta, \omega) &= \int_{-\infty}^{\infty} v^0(t - t_0) [\gamma \cos \omega t - \delta v^0(t - t_0)] dt \\
&= -\sqrt{2}\gamma \int_{-\infty}^{\infty} \operatorname{sech}(t - t_0) \tanh(t - t_0) \cos \omega t dt \\
&\quad - 2\delta \int_{-\infty}^{\infty} \operatorname{sech}^2(t - t_0) \tanh^2(t - t_0) dt. \tag{4.10}
\end{aligned}$$

The integrals are easily evaluated (the first by the method of residues) to yield

$$M(t_0, \gamma, \delta, \omega) = -\frac{4\delta}{3} - \sqrt{2}\gamma\pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin \omega t_0. \tag{4.11}$$

If we define

$$R^0(\omega) = \frac{4 \cosh\left(\frac{\pi\omega}{2}\right)}{3\sqrt{2}\pi\omega}, \tag{4.12}$$

then it follows from Theorem 3.3 that if  $\gamma/\delta > R^0(\omega)$ ,  $W^S(P_\epsilon)$  intersects  $W^U(P_\epsilon)$  for  $\epsilon$  sufficiently small and if  $\gamma/\delta < R^0(\omega)$ ,  $W^S(P_\epsilon) \cap W^U(P_\epsilon) = \emptyset$ . Moreover, since  $M(t_0, \gamma, \delta, \omega)$  has quadratic zeros when  $\gamma/\delta = R^0(\omega)$ , it follows from Corollary 3.4 that there is a bifurcation curve in the  $\gamma, \delta$  plane for each fixed  $\omega$  tangent to  $\gamma = R^0(\omega)\delta$  at  $\gamma = \delta = 0$  on which quadratic homoclinic tangencies occur. We display some calculated bifurcation curves in Figure 4.1, below.

We next compute the Melnikov function for the resonant periodic orbits. We will only consider those within  $\Gamma_+^0, q_+^k(t - t_0)$ . The resonance condition is, from (4.3)

$$2K(k)\sqrt{2 - k^2} = \frac{2\pi m}{\omega n} \tag{4.13}$$

and for each choice of  $m, n$  with  $\frac{2\pi m}{\omega n} > \sqrt{2}\pi$  (4.13) can be solved to give a unique resonant orbit  $q_+^{k(m,n)}$ . Computing

$$M^{m/n}(t_0, \gamma, \delta, \omega) = \int_{-\infty}^{\infty} v^{k(m,n)}(t - t_0) [\gamma \cos \omega t - \delta v^{k(m,n)}(t - t_0)] dt. \quad (4.14)$$

Using the Fourier series for (4.4) and Remark 3 following Theorem 3.3, we obtain

$$M^{m/n}(t_0, \gamma, \delta, \omega) = -\delta J_1(m, n) - \gamma J_2(m, n, \omega) \sin \omega t_0, \quad (4.15)$$

where  $J_1(m, n) = \frac{2}{3}[(2 - k^2(m, n))2E(k(m, n)) - 4k'^2(m, n)K(k(m, n))]/(2 - k^2(m, n))^{3/2}$ ,  
and

$$J_2(m, n, \omega) = \begin{cases} 0; & n \neq 1 \\ \sqrt{2}\pi\omega \operatorname{sech} \frac{(\pi n K(k(m, 1)))}{K(k(m, 1))}; & n = 1. \end{cases}$$

Here  $E(k)$  is the complete elliptic integral of the second kind and  $k'$  is the complementary elliptic modulus  $k'^2 = 1 - k^2$ . Defining

$$R^m(\omega) = \frac{J_1(m, 1)}{J_2(m, 1, \omega)}, \quad (4.16)$$

we conclude from Theorem 3.6 and Corollary 3.7 that if  $\gamma/\delta > R^m(\omega)$  then there is a pair of subharmonics of order  $m$  (period  $2\pi m/\omega$ ) which appear on a bifurcation curve tangent to  $\gamma = R^m(\omega)\delta$  at  $\gamma = \delta = 0$ .

Routine computations verify that

$$\lim_{m \rightarrow \infty} M^{m/1}(t_0, \gamma, \delta, \omega) = M(t_0, \gamma, \delta, \omega), \quad (4.17)$$

that the limit is approached from below and that the rate of convergence is extremely rapid (cf. Figure 6.8).

Similar computations for the orbits lying outside  $r_+^0 \cup \{(0, 0)\} \cup r_-^0$  give the Melnikov function

$$\hat{M}^{m/n}(t_0, \gamma, \delta, \omega) = -\delta \hat{J}_1(m, n) - \gamma \hat{J}_2(m, n, \omega) \sin \omega t_0 \quad (4.18)$$

with

$$\hat{J}_1(m, n) = \frac{2}{3} [(2k^2(m, n) - 1)4E(k(m, n)) + 4k'(m, n)^2 K(k(m, n))] / (2k^2(m, n) - 1)^{3/2}$$

and

$$\hat{J}_1(m, n, \omega) = \begin{cases} 0; & n \neq 1, m \text{ even}, \\ 2\sqrt{2}\pi\omega \operatorname{sech}\left(\frac{m\pi k'(m, 1)}{K(m, 1)}\right); & n = 1, m \text{ odd}. \end{cases}$$

In this case we obtain a sequence of bifurcation curves

$$\gamma = \hat{J}_1(m, 1)\delta / \hat{J}_2(m, 1, \omega) \stackrel{\text{def}}{=} \hat{R}^m(\omega)\delta \text{ accumulating on } \gamma = R^0(\omega)\delta \text{ from above.}$$

Here  $k(m, n)$  is the unique solution of the resonance relation

$$\hat{T}_k = 4K(k)\sqrt{2k^2 - 1} = \frac{2\pi m}{\omega n} \quad (4.19)$$

For more details on the above computations see Greenspan [1981].

In Figure 4.1 we show some of the approximate bifurcation curves  $\gamma/\delta = R^0(\omega), R^m(\omega)$  for orbits both inside and outside the separatrix. Below, in Figure 6.8 we show how the subharmonic bifurcation curves accumulate on the homoclinic bifurcation curve as  $m \rightarrow \infty$  for fixed  $\omega$ . We leave the discussion of the detailed structure of the Poincaré maps for various parameter values  $(\gamma, \delta, \omega)$  to Section 6.



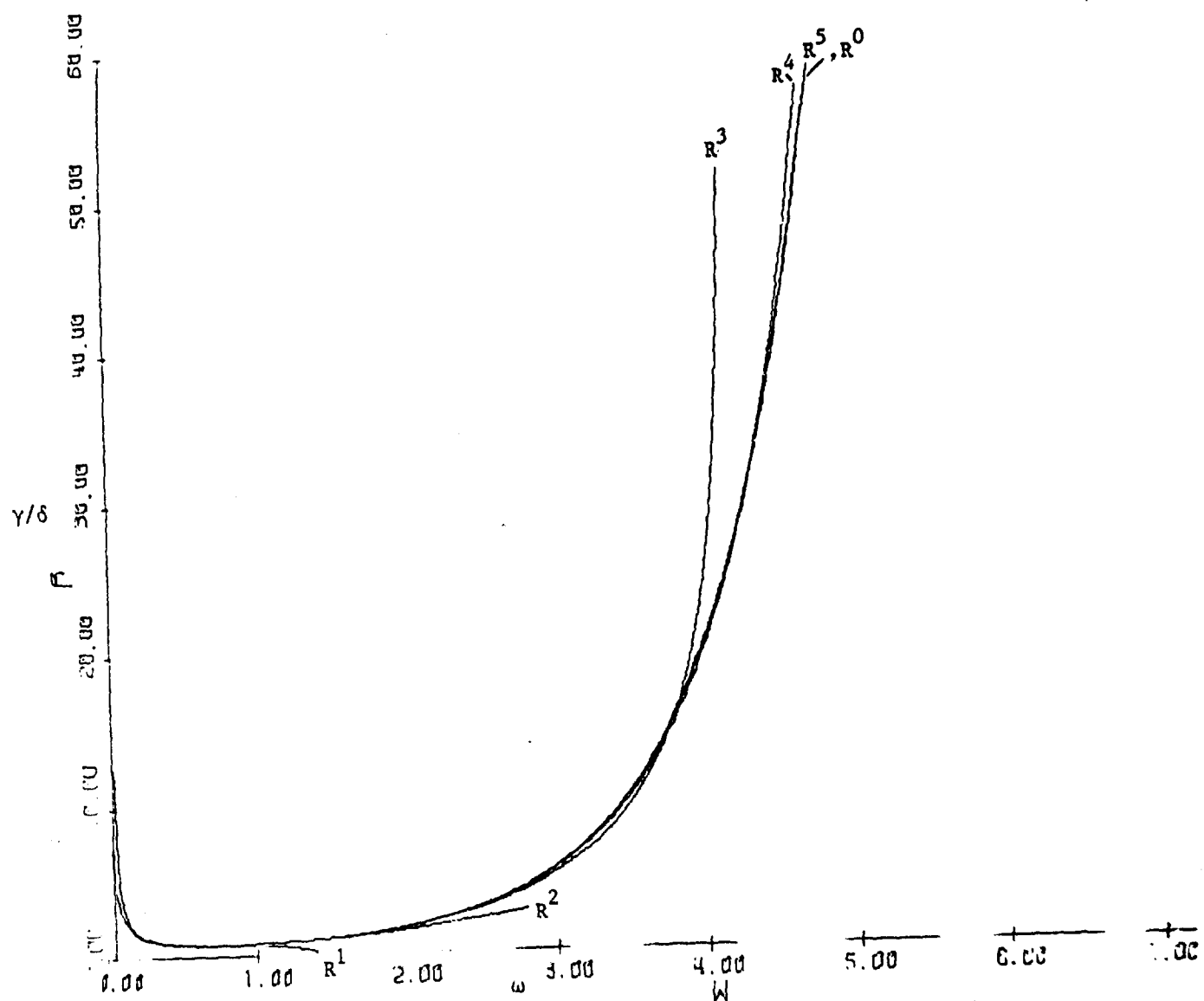


Figure 4.1 (a) Bifurcation curves  $R^m(\omega)$  for subharmonics and homoclinic orbits inside  $\Gamma_0 + U\{0,0\} \cup \Gamma_0^-$ .  
 $R^1$  = subharmonic of order 1,  $R^0$  = homoclinic orbit.

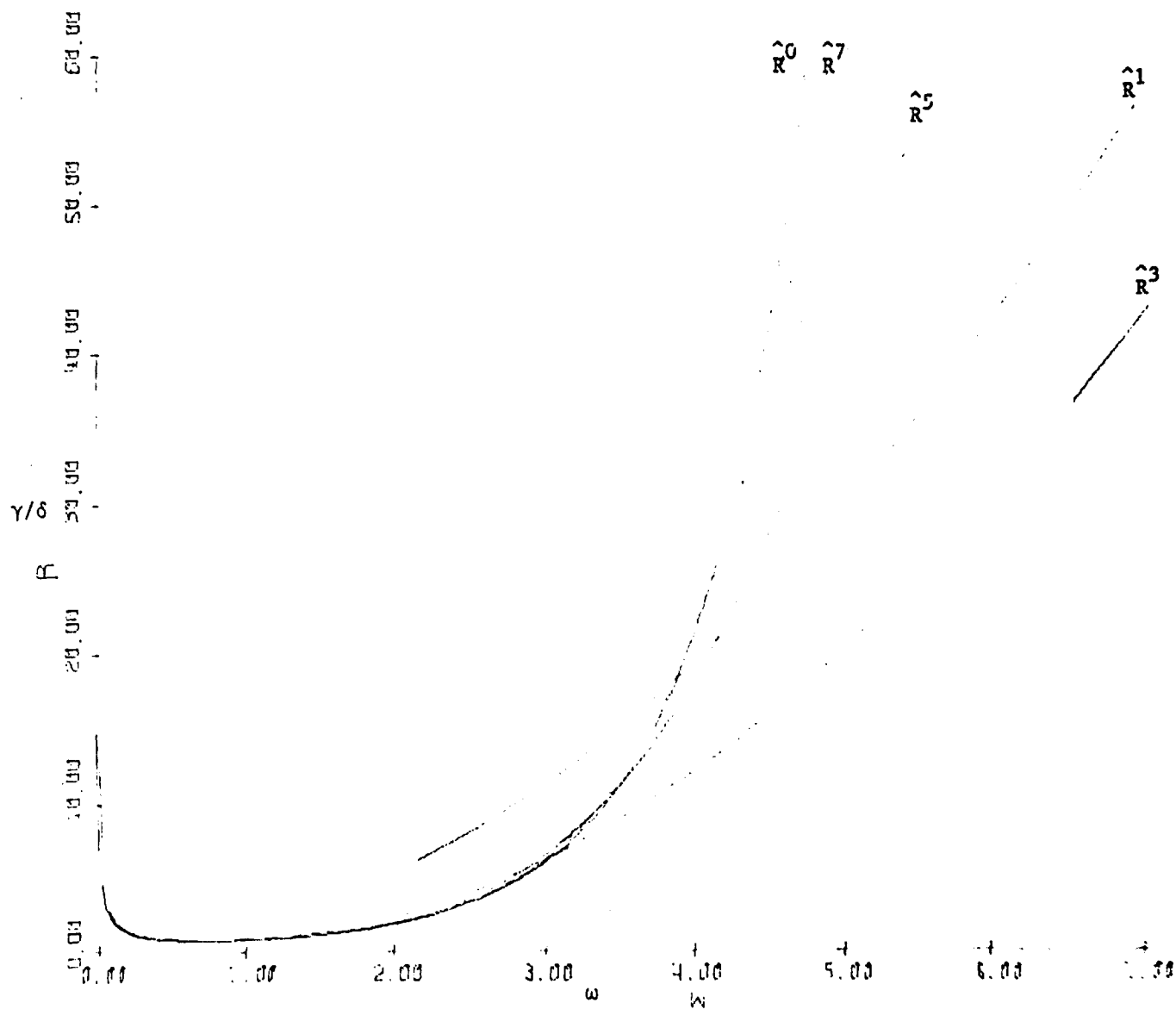


Figure 4.1 (b) Bifurcation curves  $\hat{R}^m(\omega)$  for subharmonics and homoclinic orbits outside  $\Gamma_0 + U\{0,0\} \cup \Gamma_0^-$ .

## 5. SMALE HORSESHOES, NEWHOUSE SINKS AND CHAOTIC MOTIONS

In this section we provide a brief review of some useful results in the abstract theory of dynamical systems, and in particular of two dimensional diffeomorphisms. For background material see Smale [1963,1967], Nitecki [1971], Moser [1973] or Newhouse [1980].

### 5.1 The Smale Horseshoe

Here we outline an important example of a planar diffeomorphism due to Smale [1963]. We shall adapt the map to our specific application.

Consider a map  $F: Q \rightarrow \mathbb{R}^2$  defined on the square  $Q = [0,1] \times [0,1]$  such that  $F$  is linear on the two horizontal strips  $H_1 = [0,1] \times [0,\alpha]$ ,  $H_2 = [0,1] \times [1-\alpha,\alpha]$ , with

$$DF(p) = \begin{bmatrix} \beta & 0 \\ 0 & 1/\alpha \end{bmatrix}; p \in H_1 \quad \text{and} \quad DF(p) = \begin{bmatrix} -\beta & 0 \\ 0 & -1/\alpha \end{bmatrix}; p \in H_2, \quad (5.1)$$

where  $0 < \beta < \alpha < 1/2$ .  $F$  is chosen such that the image of the strip  $[0,1] \times (\alpha, 1-\alpha)$  falls outside  $Q$  but  $F(H_1) \stackrel{\text{def}}{=} V_1$  and  $F(H_2) \stackrel{\text{def}}{=} V_2$  are two disjoint vertical strips lying in  $Q$  as shown in Figure 5.1. By our construction,  $F$  has a hyperbolic saddle point  $(0,0)$  and  $F/Q$  has constant Jacobian  $\beta/\alpha < 1$  and in thus a dissipative map (Newhouse [1979]). The nonwandering set  $\Lambda \subset Q$  of  $F$  is obtained by intersecting all forward and backward iterates:

$$\Lambda = \Lambda_V \cap \Lambda_h; \quad \Lambda_V = \bigcap_{n \geq 0} F^n(Q), \quad \Lambda_h = \bigcap_{n \geq 0} F^{-n}(Q). \quad (5.2)$$

Thus  $\Lambda_V^1 = Q \cap F(Q) = V_1 \cup V_2$  is a pair of vertical strips each of width  $\beta$ . It is not hard to see that  $\Lambda_V^k = \bigcap_{n=0}^k F^n(Q)$  is  $2^k$  disjoint vertical

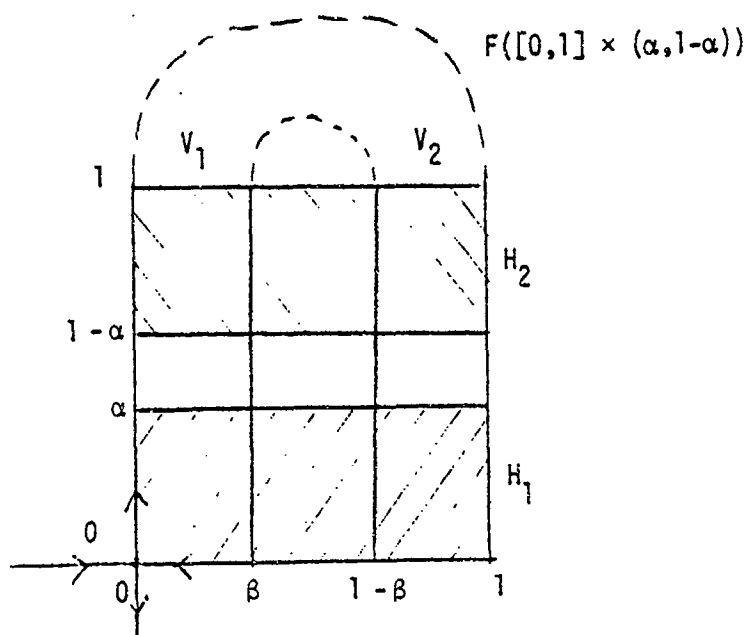


Figure 5.1. The Horseshoe

strips each of width  $\beta^k$  and thus that  $\Lambda_V = C_\alpha \times [0,1]$  is a Cantor set of vertical intervals. Similarly  $\Lambda_h^k = \bigcap_{n=0}^k F^{-n}(Q)$  is  $2^k$  disjoint horizontal strips each of width  $\alpha^k$  and  $\Lambda_k = [0,1] \times C_\alpha$  is a Cantor set of horizontal intervals. Thus  $\Lambda = C_\beta \times C_\alpha$  is also a Cantor set.

To describe the orbits of  $F|_\Lambda$  we assign to each point  $x \in \Lambda$  a bi-infinite symbol sequence  $\{a_j(x)\} = \{\dots a_{-2}a_{-1} \cdot a_0 a_1 a_2 \dots\}$  chosen such that  $a_j(x) = 1$  (resp. 2) if  $F^j(x) \in H_1$  (resp.  $H_2$ ) and  $a_{-j}(x) = 1$  (resp. 2) if  $F^{-(j-1)}(x) \in V_1$  (resp.  $V_2$ ). The action of  $F$  on  $\Lambda$  then corresponds to the shift  $\sigma$  on the space  $\Sigma$  of all such symbol sequences, and in fact there is a homeomorphism  $\phi$  such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ \downarrow \phi & & \downarrow \phi \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

To every periodic sequence there corresponds a periodic orbit of  $F|_{\Lambda}$ ; for example the fixed point  $(0,0)$  has a symbol sequence  $\{.. 111 \cdot 111 \dots\}$  and there is a second fixed point with sequence  $\{... 222 \cdot 222 \dots\}$ . For details see Moser [1973]. This method of symbolic dynamics enables us to prove the following.

Proposition 5.1. The map  $F$  has a countable set of periodic points, with points of arbitrarily high period and an uncountable set of non periodic recurrent motions. The periodic points are dense in  $\Lambda$  and there is an orbit dense in  $\Lambda$ . All the periodic points are of saddle type.

The last assertion follows directly from the form of the linearized map  $DF$ . It is more difficult to prove that  $F|_Q$  is structurally stable, but this latter important fact implies that any sufficiently  $C^1$ -close map  $\hat{F}$  also possesses an invariant cantor set  $\hat{\Lambda}$  homeomorphic to  $\Lambda$ . Thus the complicated dynamics of  $F$  cannot be removed by small perturbations.

The 'piecewise linear' map  $F$  represents an idealization of a map which is embedded in the dynamics of any system with transverse homoclinic orbits. This result, the Smale-Birkhoff homoclinic theorem (cf. Birkhoff [1927] Smale [1963], Moser [1973]) follows from the lambda lemma and the homomorphic structure of Figure 2.2, although it can be proved independently. We select a small rectangle  $S$  bounded by pieces of the stable and unstable manifolds  $W^S(p), W^U(p)$ , with a transverse homoclinic point  $q$  at one corner. There are then integers  $\ell_1, \ell_2, n_i$  such that  $A = P^{-\ell_1}(S)$ ,  $B = P^{\ell_2}(S)$  are disjoint and lie in any neighborhood  $U_\nu(p)$ , and the map  $P^{n_i}: B \rightarrow A$  is well approximated by its linearization  $DP^{n_i}(p)$ . This enables one to find horizontal strips  $H_1, H_2 \subset B$  whose images under

$p^{l_1+l_2+n_i}$  are vertical strips  $V_1, V_2 \subset B$ . The map  $p^{(l_1+l_2+n_1) \cdot (l_1+l_2+n_2)}$  then has a horseshoe. A careful study of the linearized map enables one to obtain estimates of hyperbolicity and show that the behavior of  $p^{(l_1+l_2+n_1)(l_1+l_2+n_1)}$  is homeomorphic to that of the 'idealized' map  $F$ . Specifically, for our problem we have

Theorem 5.2. If the Poincaré map  $p_\epsilon^{t_0} : \Sigma^{t_0} \rightarrow \Sigma^{t_0}$  possesses a transverse homoclinic point  $q_\epsilon$  to a hyperbolic saddle point  $p_\epsilon$ , then, in a neighborhood of  $q_\epsilon$ , some power  $(p_\epsilon^{t_0})^N$  possess an invariant zero-dimensional cantor set  $\Lambda$  on which  $(p_\epsilon^{t_0})^N$  is conjugate to a shift on two symbols.

Corollary 5.3.  $(p_\epsilon^{t_0})^N / \Lambda$  possess a dense set of periodic points, there are points of arbitrarily high period and there is a non-periodic orbit dense in  $\Lambda$ . Moreover  $\Lambda$  is structurally stable in the sense that if  $\hat{P}$  is sufficiently  $C^1$  close to  $p_\epsilon^{t_0}$ , then  $(\hat{P})^N$  has a non-wandering set  $\Lambda$  and there is a homeomorphism  $h$  such that  $(p_\epsilon^{t_0})^N \circ h = L \circ \hat{P}^N$ .

The horseshoe's non-wandering set  $\Lambda$  is extremely complicated. It is of saddle type and has one dimensional stable and unstable bundles  $W^S(\Lambda)$ ,  $W^U(\Lambda)$  which are locally the products of Cantor sets and curves. Letting  $W^S(0,0)$ ,  $W^U(0,0)$  be the stable and unstable manifolds of the fixed point  $\{...111 \cdot 111...\}$  of the idealized horseshoe,  $F$ , we have  $W^S(\Lambda) = C\ell(W^S(0,0))$  and  $W^U(\Lambda) = C\ell(W^U(0,0))$ , and since such a  $\Lambda$  can be chosen arbitrarily close to any transverse homoclinic point  $q_\epsilon$  and hence to the saddle point  $p_\epsilon$  in our application, it follows that

$$W^S(\Lambda) = C\ell(W^S(p_\epsilon))$$

$$W^U(\Lambda) = C\ell(W^U(p_\epsilon)).$$

Thus the structure of the stable and unstable manifolds  $W^S(p_e), W^U(p_e)$  is doubly important not just for establishing the existence of horseshoes, but also in their structure. We shall return to this in Section 6.

## 5.2 Homoclinic Tangencies and Newhouse Sinks

We now turn to the bifurcations in which horseshoes are created. Newhouse [1974, 1979, 1980] has studied the situation and has shown that the lack of hyperbolicity which occurs when stable and unstable manifolds have a quadratic tangency persists under small perturbations. We first require a definition. A hyperbolic (basic) set  $\Lambda(F)$  for a two dimensional diffeomorphism  $F$  is called wild if there is a  $C^r$ -neighborhood  $N$  of  $F$  such that for any  $G \in N$  there are points  $x, y \in \Lambda(G)$  such that  $W^U(x)$  and  $W^S(y)$  are tangent somewhere.

Theorem 5.4. (Newhouse [1979]). Let  $p$  be a dissipative saddle point of a two dimensional  $C^r$  diffeomorphism  $F$  ( $|\det(DF(p))| < 1$ ). Suppose  $W^U(p), W^S(p)$  are tangent at some point  $q$ . Then arbitrarily  $C^r$  near  $F$  there is a diffeomorphism  $G$  having a wild hyperbolic set near the orbit of  $q$ .

Wild hyperbolic sets are important because they imply the existence of countably many stable periodic orbits.

Theorem 5.5. Suppose  $F$  is a two dimensional  $C^r$  diffeomorphism with a wild hyperbolic set  $\Lambda$  containing a dissipative periodic point. Then there is a neighborhood  $N$  of  $F$  and a residual subset  $B \subset N$  such that if  $G \in B$ , then  $G$  has countable many periodic sinks.

For a proof of Theorem 5.5 see Newhouse [1980]. One step in the proof closely resembles a result obtained independently by Gavrilov and Gilnikov [1972, 1973], who showed that, for certain two dimensional diffeomorphisms, the homoclinic tangencies were the limit of an infinite sequence of saddle-node bifurcations in which pairs of periodic sinks and saddles are created. To illustrate their argument, consider a one parameter family of maps  $F_\mu$  such that for  $\mu < 0$   $W^u(p) \cap W^s(p) = \emptyset$  while for  $\mu > 0$  we have transversal intersections and at  $\mu = 0$   $F_0$  has a quadratic tangency as in Theorem 5.4. We pick a coordinate system  $(x, y)$  such that, in a neighborhood  $U_\nu(p)$ ,  $W^s(p)$  and  $W^u(p)$  are given by  $y = 0$  and  $x = 0$  respectively. Let  $q_2 = (0, \bar{y})$  and  $q_1 = (\bar{x}, 0) = F_0^n(q_2)$  be (non-transverse) homoclinic points and  $A_1, A_2$  be neighborhoods of  $q_1, q_2$  lying in  $U_\nu$  as shown in Figure 5.2. The map  $F^k : A_1 \rightarrow A_2$  is well approximated by

$$DF(p))^k = \begin{bmatrix} \lambda^k & 0 \\ 0 & \gamma^k \end{bmatrix}, \quad (5.3)$$

where  $0 < \lambda < 1 < \gamma$  and  $\lambda\gamma < 1$  and  $k \rightarrow \infty$  as we pick domains in  $A_1$  closer and closer to  $W^s(p)$ . The form of the map  $F_0^n : A_2 \rightarrow A_1$  ( $n$  is fixed) may also be approximated since, by hypothesis, the vertical line  $W^u(p) \cap A_2$  is mapped to the parabola  $W^u(p) \cap A_1$  and the parabola  $W^s(p) \cap A_2$  to the horizontal line  $W^s(p) \cap A_1$ . When  $\mu \neq 0$  the images also undergo a vertical translation. Using these facts, and normalizing the maps to remove redundant parameters, we arrive at an approximate quadratic map

$$F_\mu^{m+k} \approx f_{\mu,k} : (x - \bar{x}, y) \rightarrow (-(\gamma^k y - \bar{y}), \lambda^k x + a(\gamma^k y - \bar{y})^2 - \mu). \quad (5.4)$$

(We have selected the orientation such that  $a > 0$ ). It is interesting to note that this is essentially the same quadratic map studied by Hénon [1976]





bifurcations. For sufficiently large  $k$ , these subharmonics of order  $m \stackrel{\text{def}}{=} n + k$  are precisely those found by Melnikov's method in Sections 3 and 4, since the perturbation methods used there yield all subharmonics which pass once through  $U_v$ .

The two fixed points of (5.4) existing for  $\mu > \mu_k$  are a saddle and a sink and another straightforward analysis shows that the sink bifurcates to a sink of period two at

$$\mu'_k = \lambda^k \bar{x} - \gamma^{-k} \bar{y} + \frac{3}{4a} (\lambda^k + \gamma^{-k})^2, \quad (5.6)$$

leaving to an orbit of period  $2m$  for the map. The bifurcation sequence  $\mu'_k$  also accumulates on  $\mu = 0$  from below. Recent work of Lanford [1981] has shown that quadratic maps of the plane with small Jacobian determinant undergo countable sequences of period doubling bifurcations precisely as do one dimensional maps to which they are sufficiently close (Feigenbaum [1978], Collet, Eckmann and Lanford [1980]). The present map has  $|Dg_{v,k}| = (\lambda\gamma)^k$  and therefore falls into this class for  $k$  large. We conclude that  $\mu'_k$  is but the first of a countable set of period doubling bifurcations which must take place before  $\mu = 0$ , since for  $\mu > 0$  there is a full horseshoe in  $A_1$  near  $q_1$ , containing points of all periods for the map  $F^{n+k}$ . In fact countable sets of further bifurcations must also occur, in which points of period  $2^j \cdot 3(n+k)$ ,  $2^j \cdot 5(n+k)$ ;  $j = 1, 2, \dots$  etc. are created. Since such sequences occur for each sufficiently large  $k$  we have a countable sequence of countable sequences of bifurcations all accumulating on  $\mu = 0$ . One can thus find sinks of arbitrarily high period in any neighborhood of a non transverse homoclinic point  $q$ , and finite sets of these sinks persist when the homoclinic tangency is broken. Newhouse's results [1979, 1980] give more information and in particular show that, in the case of tangencies, the

closure of the set of homoclinic points of a suitably perturbed map is contained in the closure of the set of periodic sinks. This implies that any homoclinic orbit can be approximated arbitrarily closely (for finite, but arbitrarily long times) by a stable periodic motion and thus the horseshoe might become 'physically observable'. On the basis of this work Newhouse [1979] conjectured that some of the numerically observed strange attractors (such as that of Hénon [1976]) might in fact be orbits lying in the stable manifolds of long period sinks. The question is still open, and we shall return to it in the next section.

Before closing we note that, even without Theorem 5.4, the existence of a single parameter value  $\mu$  at which a homoclinic tangency occurs immediately implies a countable set of such values, since the stable and unstable manifolds accumulate on one another, by the lambda lemma. Thus the 'first' tangency is immediately followed by infinitely many others, each with their attendant saddle-node and period doubling bifurcations.

## 6. THE GLOBAL STRUCTURE OF SOLUTIONS OF DUFFING'S EQUATION

In this final section we make use of the bifurcation and perturbation computations of Sections 3 and 4 and the abstract theory of Section 5 to obtain a partial understanding of Duffing's equation (4.1) for  $\epsilon$  small. We start by considering the phase portraits of the averaged systems (3.29) and (3.33) near each resonant periodic orbit inside the separatrix  $\Gamma_+^0$ . Similar conclusions apply within  $\Gamma_-^0$  and outside the appendix.

### 6.1 Subharmonics and Their Domains of Attraction

The material in this and the following section is deduced directly from the theory of Section 3, the results of Section 4, and standard results in dynamical systems theory such as that reviewed in Section 5. In these two sections we collect various proven facts concerning Duffing's equation, although we do not state precise theorems. In the concluding Section 6.3 we go on to make some conjectures which we feel have a firm basis in these facts.

Within  $\Gamma_+^0$  the action transformation gives

$$I^k = I(k) = \frac{1}{3\pi} [2(2 - k^2)E(k) - 4k'^2 K(k)] / (2 - k^2)^{3/2}, \quad (6.1)$$

where the elliptic modulus  $k$  depends implicitly upon  $u$  and  $v$  through the relationships  $H(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4} = \frac{k^2 - 1}{(2 - k^2)^2}$ . Using (4.13) we have

$$\begin{aligned} \Omega'(I^k) &= \frac{d}{dI} \left( \frac{2\pi}{T(I)} \right) \Big|_{I=I^k} = - \frac{2\pi}{T_k^2} \frac{dT(k)/dk}{dI(k)} \\ &= - \frac{\pi^2 (2 - k^2) [(2 - k^2)E(k) - (2 + k^2)k'^2 K(k)]}{4k^4 k'^2 K(k)^3} \equiv \Omega'(m), \end{aligned} \quad (6.2)$$

where  $k = k(m)$  is defined by the relationship  $K(k)\sqrt{2 - k^2} = \pi m/\omega$  on a resonant orbit of period  $2\pi m/\omega$  and  $k'^2 = 1 - k^2$ . Thus (3.29) and its integral (3.30) become

$$\dot{\phi} = \sqrt{\epsilon} \Omega'(m) h, \quad (6.3)$$

$$\dot{h} = \sqrt{\epsilon} \frac{1}{2\pi} (-\delta J_1(m) - \gamma J_2(m) \sin(m\phi)),$$

$$\text{and } H = \sqrt{\epsilon} \left[ \frac{\Omega'(m)}{2} h^2 + \frac{1}{2\pi} \left( \delta J_1(m) \delta\phi - \gamma \frac{J_2(m, \omega)}{m} \cos(m\phi) \right) \right], \quad (6.4)$$

where we have used equation (4.15) (with  $J_1(m, 1) \stackrel{\text{def}}{=} J_1(m)$ ) and the fact that  $\Omega(I^\alpha) = \omega/m$  at resonance. Noting that  $\delta J_1(m)$ ,  $\gamma J_2(m, \omega)$  are positive while  $\Omega'(m)$  is negative we easily sketch the phase portraits of (6.3) for the three cases  $\gamma/\delta <, =, > R^m(\omega)$ , as in Figure 6.1. Note that, for  $\gamma/\delta > R^m(\omega)$ , in each resonant band there are  $m$  saddles and  $m$  'centers' corresponding to a pair of subharmonics of order  $m$  in the original system.

To study the actual stability type of the fixed points which appear as centers in the  $O(\sqrt{\epsilon})$  approximation, we compute the averaged higher order terms  $\overline{F}$ ,  $\overline{G}$  of Eq. (3.33). After tedious but routine computations we obtain

$$\begin{aligned} \overline{F}(\phi) &= -\gamma K_2(m, \omega) \sin(m\phi) - \delta K_1(m) \\ \overline{G}(\phi) &= -\gamma K_2(m, \omega) \cos(m\phi)/m \end{aligned} \quad (6.5)$$

where  $K_1$  and  $K_2$  are <sup>positive</sup> constants found by integrating products of elliptic and trigonometric functions, much as in the first order terms computed in section four. We note that we do not need  $\Omega''$  explicitly, and that the trace (up to  $O(\epsilon)$ ) is given by

$$\frac{\partial}{\partial h} [\epsilon \overline{F}(\phi) h] + \frac{\partial}{\partial \phi} [\epsilon (\delta \Omega'' h^2 + \overline{G}(\phi))] = -\epsilon \delta K_1(m)$$

(cf. eq. (3.33)).

These fixed points are thus found to be sinks, and the bifurcations in which they are created simple saddle nodes, rather than the doubly degenerate Hamiltonian bifurcations occurring in (6.3) when  $\gamma/\delta = R^m(\omega)$ . We show the phase portraits in Figure 6.2. The stable manifolds or domains of attraction of the sinks are shaded.

The methods used here are quite standard and include a check on the non existence of closed orbits by Bendixon's criterion. Morosov [1973], [1976] obtained similar results for a related equation.

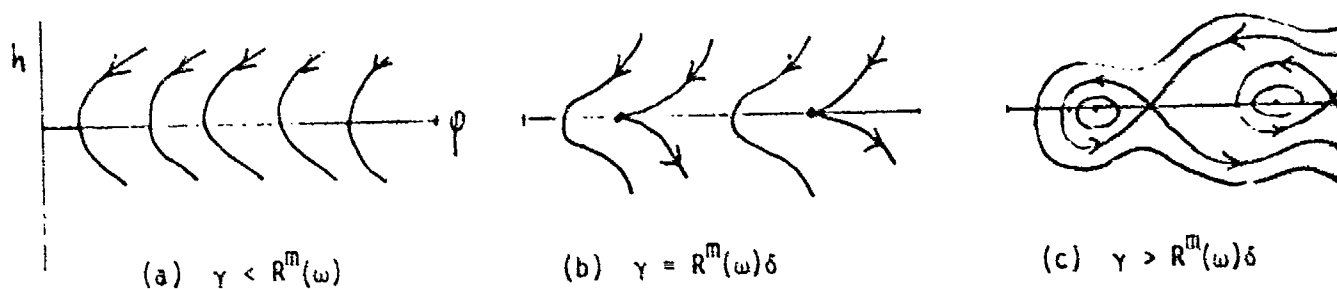


Figure 6.1. Phase portraits of Eq. 6.3,  $m = 2$ .

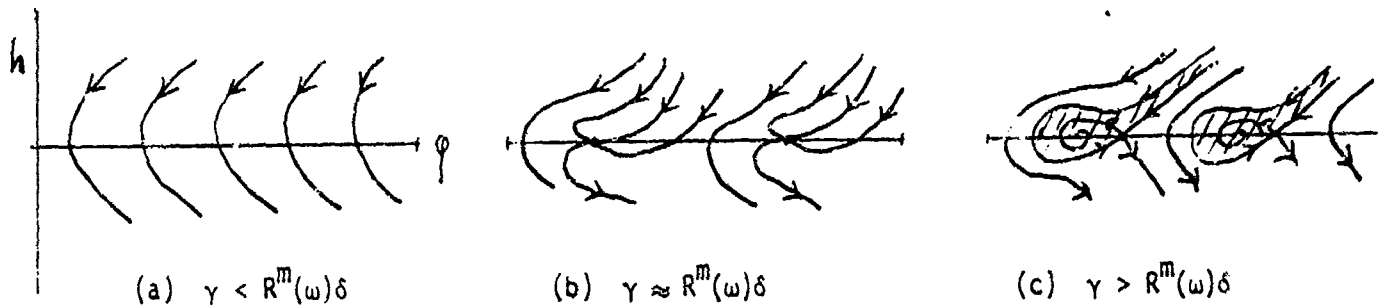


Figure 6.2. Phase portraits of Eq. 6.3 with  $O(\epsilon)$  terms added,  $m = 2$ .

It is important to realize, however, that unlike the existence Theorem 3.6 these results are not uniformly valid since the factor  $\Omega'(m)$  in (6.3) becomes unbounded as  $m \rightarrow \infty$ ; in fact using the asymptotic behavior  $K(k) \sim \ln(4/k')$  as  $k \rightarrow 1(k' \rightarrow 0)$ , we have

$$\Omega'(m) \sim \frac{-\pi^2}{4k'^2 [\ln(4/k')]^3} = -\frac{4\omega^3 e^{2\pi m/\omega}}{\pi m^3}, \quad (6.6)$$

as  $m \rightarrow \infty$ . Thus averaging is valid for successively smaller regions  $0 < \epsilon \leq \epsilon_0(m)$  as  $m$  increases, and, while the invariant manifolds and periodic points of the Poincaré map  $P_\epsilon$  are qualitatively similar to those of Figure 6.2 in any given resonance band, we cannot carry our stability results uniformly to the limit of homoclinic bifurcations as in Theorem 3.8. Our results are, however, good for subharmonics of all finite orders  $m < \infty$  and hold within each resonance band near the unperturbed resonant orbit of period  $2\pi m/\omega$ .

Noting that  $J_1(m, n) > 0$  while  $J_2(m, n, \omega) = 0$  for  $n \neq 1$  (Eq. 4.15) we conclude that, between the resonance bands all orbits decay 'inward' ( $h(t)$  decreases). This implies that the unstable manifolds  $W^u(s_k)$  of the saddles of period  $k$  intersect the stable manifolds  $W^s(s_{k-1})$  of those of period  $k-1$  and one therefore obtains the global structure of connected resonance bands indicated in Figure 6.3 (cf. Hayashi [1980], Figure 3).

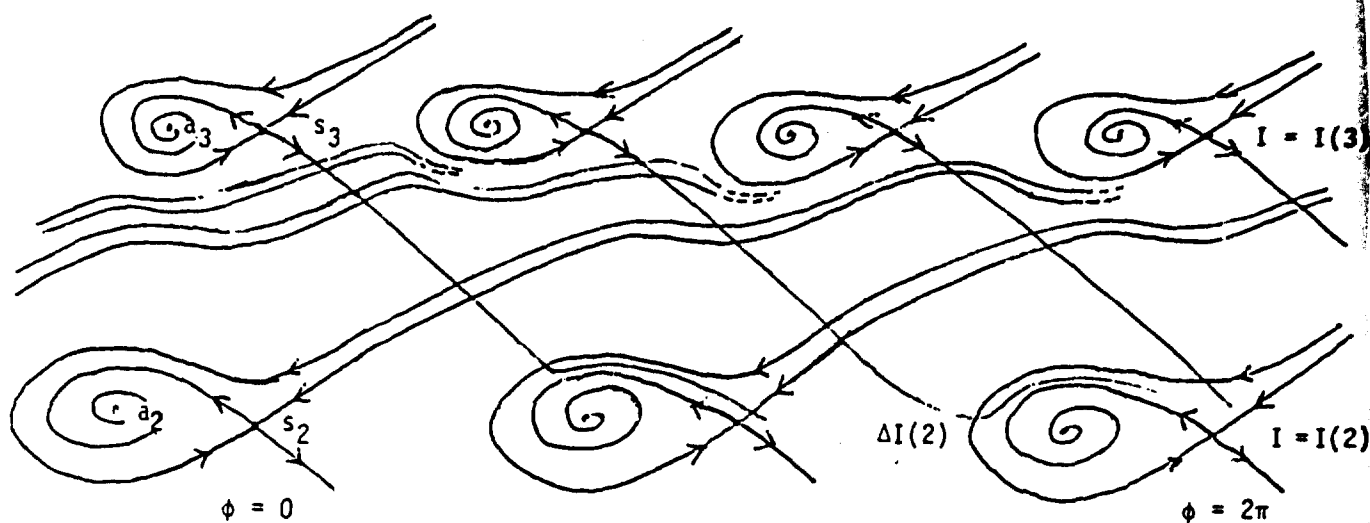


Figure 6.3 Heteroclinic orbits connecting neighboring resonance bands  $m = 3$  and  $m = 2$ .

This structure is repeated between each pair of resonance bands and, using the lambda lemma, it is possible to prove that, in any neighborhood  $U$  of the stable manifold  $W^S(s_k)$  of a saddle of period  $k$  there will be pieces of the stable manifolds  $W^S(s_j), W^S(a_j)$  of saddles and sinks of all lower periods  $j < k$ . The resultant violent winding and packing of the alternating domains of attraction of the stable subharmonics leads to a sensitive dependence on initial conditions, especially for the higher order subharmonic motions, since the basic width of the domains of attraction, calculated from the integral (6.4) of the  $O(\sqrt{\epsilon})$  Hamiltonian approximation is

$$\Delta I(m) = \left[ \frac{\epsilon}{\pi \Omega'(m)} (\gamma J_2(m) \cos(m\hat{\phi}) - \delta J_1(m)\hat{\phi}) \right]^{1/2} + O(\epsilon) \quad (6.7)$$

where  $\hat{\phi}$  is the root of

$$\sin \phi = - \frac{\delta J_1(m)}{\gamma J_2(m)} = -\delta R^m(\omega) \quad (6.8)$$

corresponding to the center. For  $m$  large, while  $J_1(m), J_2(m, \omega)$  are bounded (Theorem 3.8 and §4),  $\Omega'(m) \rightarrow \infty$  and we find



$$\Delta I(m) \sim \sqrt{\epsilon \frac{m^3}{4\omega^3}} e^{-\pi m/\omega} \quad (6.9)$$

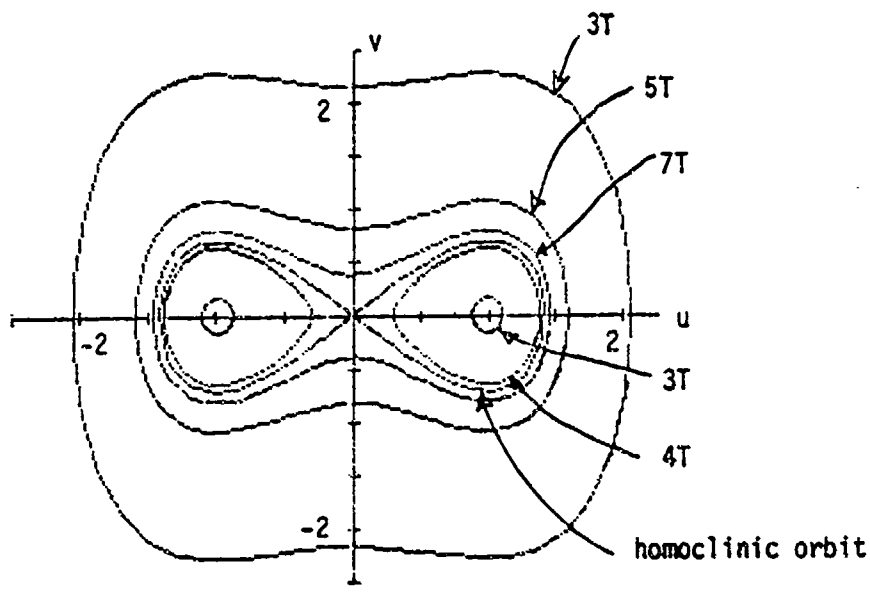
and thus the domains shrink rapidly in size as  $m$  increases. In typical computations all but the lowest order ( $m \leq 5$ , say) subharmonics lie within a band whose area is less than 1% of the interior of  $\Gamma_+^0$ !

In Figure 6.4 we illustrate our results with numerical computations of the Poincaré map, showing unperturbed resonant orbits and the perturbed subharmonics. For more detailed and extensive computations on different versions of Duffing's equation, see Hayashi [1975,1980] or Ueda [1980, 1981a]. The interaction of the subharmonics of order  $m = 2$  and  $3$  with the fixed points near  $(u,v) = (\pm 1,0)$  has been studied using conventional averaging methods by Holmes and Holmes [1981].

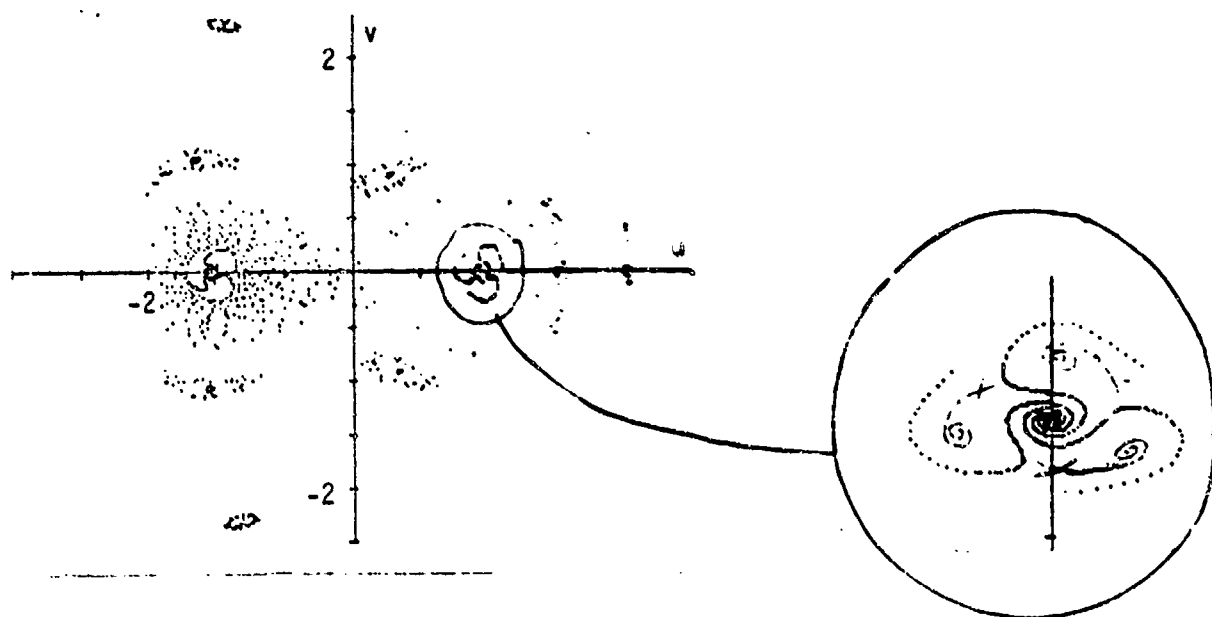
## 6.2 Homoclinic Orbits and Chaotic Motions

As we pointed out in Section 6.1, the results based on the averaged systems (3.29) and (3.33) are invalid near the homoclinic orbit, since  $|\Omega'(m)| \rightarrow \infty$  as  $m \rightarrow \infty$ . We can, however, use the abstract results of Section 5 to study the global structure of the Poincaré map in this region, and we do know from Theorems 3.6 and 3.8 and Corollary 3.7 that subharmonics of all orders exist here for  $\gamma > R^0(\omega)\delta$ , although we cannot compute their stability types from Eq. (3.33).

We start by noting that the symmetry of solutions of (4.1) (if  $(u,v,t)$  is a solution, then  $(-u,-v,t + \pi/\omega)$  is also a solution) together with the presence of transverse homoclinic points for  $\gamma/\delta > R^0(\omega)$  immediately imply that we can find strips  $S_+$ ,  $S_-$  'parallel' to  $W^S(p)$  and in a neighborhood



(a) Unperturbed resonant level curves,  $T = \frac{2\pi}{\omega}$  = period of forcing function



(b) Subharmonics of orders three and five outside  $H = 0$  and of orders one and three inside  $H = 0$ .

Figure 6.4. Some numerically computed subharmonics for Duffing's equation:  
 $\omega = 4.2$ ,  $\varepsilon\gamma = 0.65$ ,  $\varepsilon\delta = 0.005$ .

$U_V(p_\epsilon)$ , whose images  $P_\epsilon^N(S_\pm)$  under some iterate  $P_\epsilon^N$  of the Poincaré map are as shown in Figure 6.5. The orbit of any point  $x \in S_+$  (resp.  $S_-$ ) will remain near the unperturbed loop  $\Gamma_+^0$  (resp.  $\Gamma_-^0$ ) until it re-enters  $U_V$ . In this way, using the symbolic dynamics approach of Section 5.1, we can find orbits of  $P$  which visit neighborhoods of  $\Gamma_+^0$  and  $\Gamma_-^0$  in any specified order.

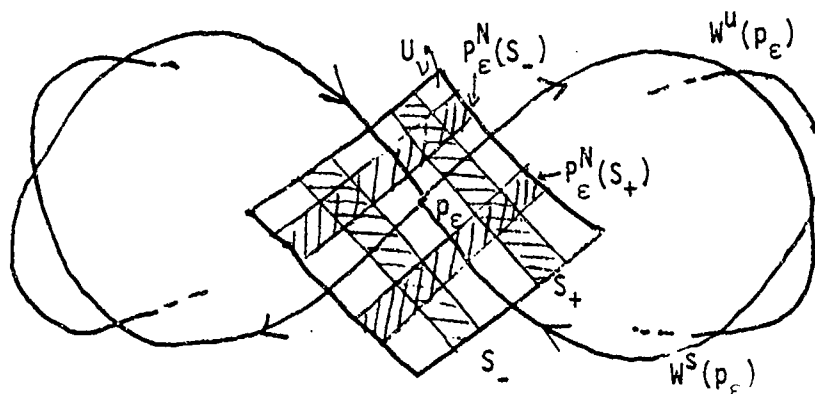


Figure 6.5 The double homoclinic structure

This goes some way toward explaining the chaotic snap through motions of Figure 1.3 (cf. Tseng and Dugundji [1971], Moon and Holmes [1979], Moon [1980]), but we must recall that all these orbits are of unstable saddle type, their dynamics being dominated by the linearized map  $DP_\epsilon(p_\epsilon)$ . They would thus be expected to give rise only to transient chaotic motions and this is, indeed, precisely what is observed when numerical integrations are performed in certain parameter ranges; almost all orbits converging to fixed points or subharmonics lying within  $\Gamma_+^0$  or  $\Gamma_-^0$  (cf. Figure 6.4). However, in view of Newhouse's results on persistent tangencies and wild hyperbolic sets, it is reasonable to expect that there exist stable long period motions which approximate such saddle-type chaotic orbits arbitrarily closely on finite time intervals. We return to this point in Section 6.3.

Note that while such 'chaotic subharmonics' cannot be found by the perturbation procedures of Section 3, since they pass through  $U_V(p_\epsilon)$  arbitrarily often, their existence can be inferred from our perturbation calculations of  $W^S(p_\epsilon)$  and  $W^U(p_\epsilon)$ .

We also note that, by arguments similar to those of Section 6.1, we can conclude that pieces of the unstable manifold  $W^U(p_\epsilon)$  accumulate on all the unstable manifolds  $W^U(s_k)$ ,  $k = 1, 2, \dots$  of all periodic saddle type motions within  $\Gamma_+^0$  and  $\Gamma_-^0$ . It follows that any attracting periodic motions within this region lie in  $Cl(W^U(p_\epsilon))$ .

In earlier work (Holmes [1979]) we showed that all orbits of the damped Duffing equation remain bounded and enter some bounded simply connected set  $A \times S^1 \subset \mathbb{R}^2 \times S^1$  as  $t \rightarrow \infty$ . Letting  $A \in \Sigma^0$  be a section of such a set, if we define the attracting set  $A$  as

$$A = \bigcap_{n \geq 0} (P_\epsilon)^n(A); \quad (6.10)$$

it follows that

$$A = Cl(W^U(p_\epsilon)). \quad (6.11)$$

Here we assume that any attracting sinks  $\epsilon$  lying inside  $A$  but outside  $\Gamma_\pm^0$  are sufficiently close so that pieces of  $W^U(p_\epsilon)$  lie in their stable manifolds. However, as we show in the next section, it is not likely that  $A$  is a 'nice'

attractor; an invariant attracting set which is indecomposable and contains a dense orbit. Certainly for parameter values  $(\gamma, \delta, \omega)$  for which tangencies occur  $Cl(W^U(p_\epsilon))$  contains periodic sinks and hence is decomposable.

### 6.3 The Strange Attractor: Numerical Results and Conjectures

We start by summarizing the results of numerical integrations and computations of the Poincaré map, drawing in the results of Holmes [1979], Shaw [1980] and, principally, Ueda [1981b]. In Figure 6.6a-d we show plots of

the stable and unstable manifolds  $W^S(p_\epsilon)$ ,  $W^U(p_\epsilon)$  for fixed  $\epsilon\delta = 0.25$ ,  $\omega = 1.0$  and  $\epsilon\gamma$  varying. These curves are obtained by iterating short line segments lying near  $W_{loc}^S(p_\epsilon)$ ,  $W_{loc}^U(p_\epsilon)$ . We note that the manifolds have just intersected for the first time at  $\epsilon\gamma = 0.19$  the theoretical value from Eq. (4.12) being  $\epsilon\gamma = 0.188$ . As  $\gamma$  is increased for fixed  $\delta$  the homoclinic bifurcation and resulting formation of horseshoes near  $W^U(p_\epsilon)$  occurs while stable fixed points continue to exist within each of the (unperturbed) loops  $\Gamma_+^0$ ,  $\Gamma_-^0$ . However, with appreciably higher  $\gamma$  and for a wide range of  $\gamma, \delta$  values, numerical computations suggest that there are no stable low period attractors and one observes the characteristic 'strange attractor'\* motions of Figure 6.7. Comparing Figure 6.7a with 6.6d, we note that this single orbit appears to fall on  $C \cap W^U(p_\epsilon)$ , as predicted in Section 6.2. For typical power spectra of these motions see Holmes [1979] or Ueda [1980, 1981a].

In Figure 6.8 we indicate regions in  $\epsilon\gamma, \epsilon\delta$  parameter space for  $\omega = 1$  in which the various attractors are observed. We also superpose some bifurcation curves from our theory. In each region we sketch a representative motion projected onto  $u-v$  space. These numerical results are due to Ueda [1981b]. Note how the subharmonic bifurcations all occur very close to the homoclinic bifurcation curve  $\gamma = R^0(\omega)\delta$ , apart from those of order  $m = 1$ . Above  $\gamma = R^1(\omega)\delta$  subharmonics of all orders exist both inside and outside the unperturbed separatrix  $\Gamma_0^+ \cup \{p_0\} \cup \Gamma_0^-$ , although only low period ones are observable. The theoretical value of  $\hat{R}^1$  agrees well with that observed.

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\* This term is used only for convenience; we should more correctly say 'apparent non-periodic attracting motion'. As indicated at the end of Section 6.2, we define a strange attractor for the map  $P$  as an attracting set  $A$  which is indecomposable, has a dense orbit and is neither a fixed point nor a periodic orbit.

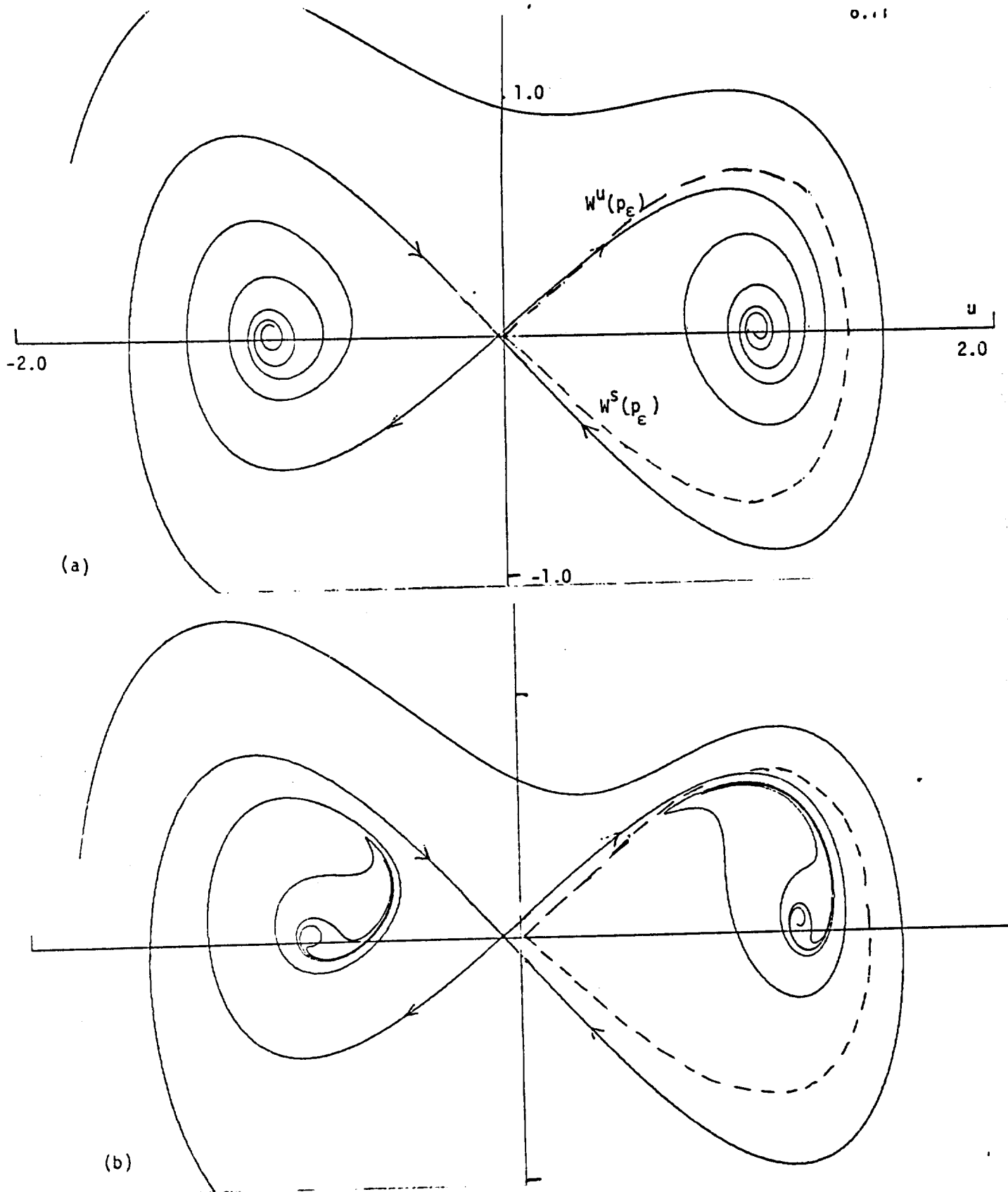


Figure 6.6. Numerically computed Poincaré maps for Duffing's equation.  $\omega = 1.0$ ,  $\epsilon\delta = 0.25$ . (a)  $\epsilon\gamma = 0.05$ , (b)  $\epsilon\gamma = 0.15$ . Perturbed manifolds —; unperturbed manifolds - - - -.

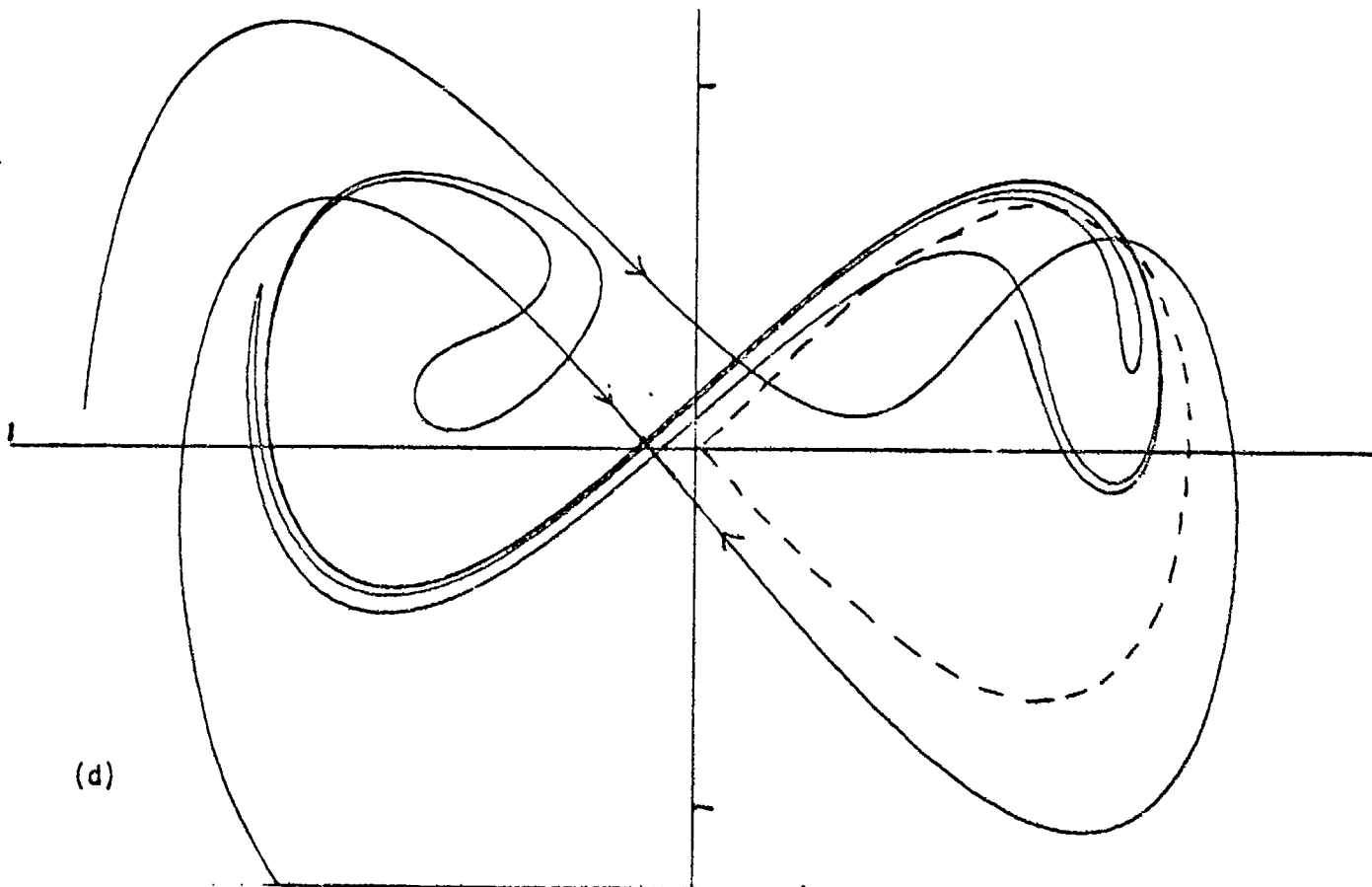
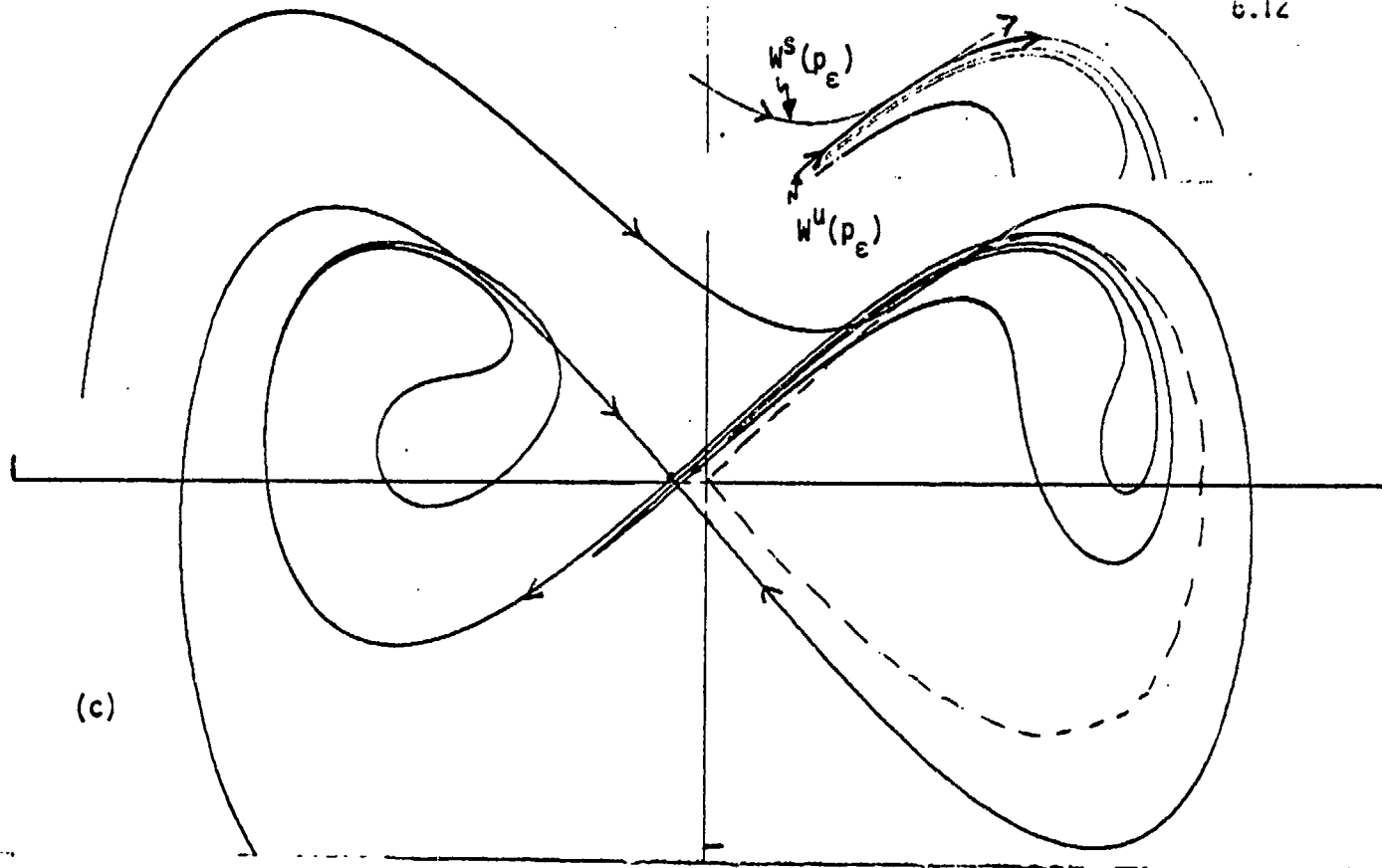
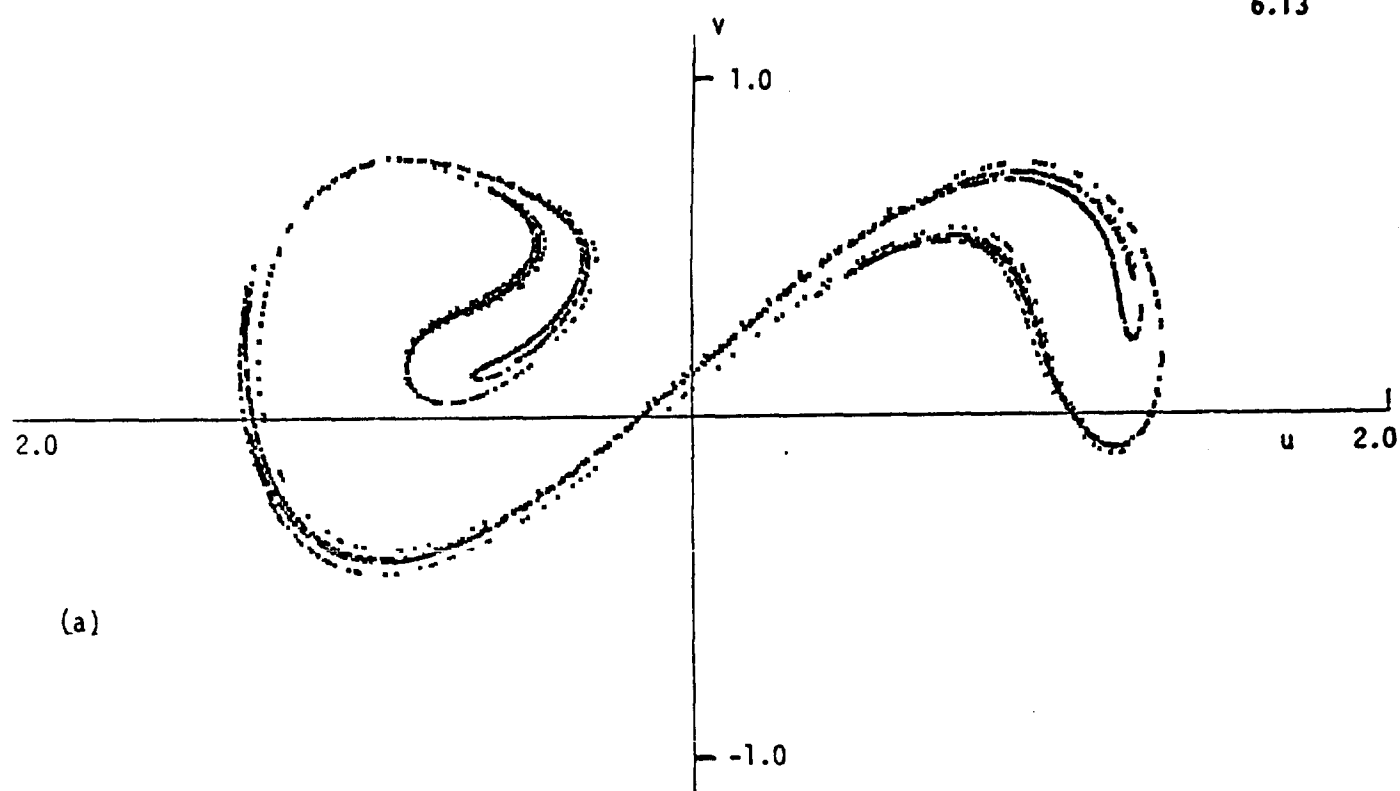
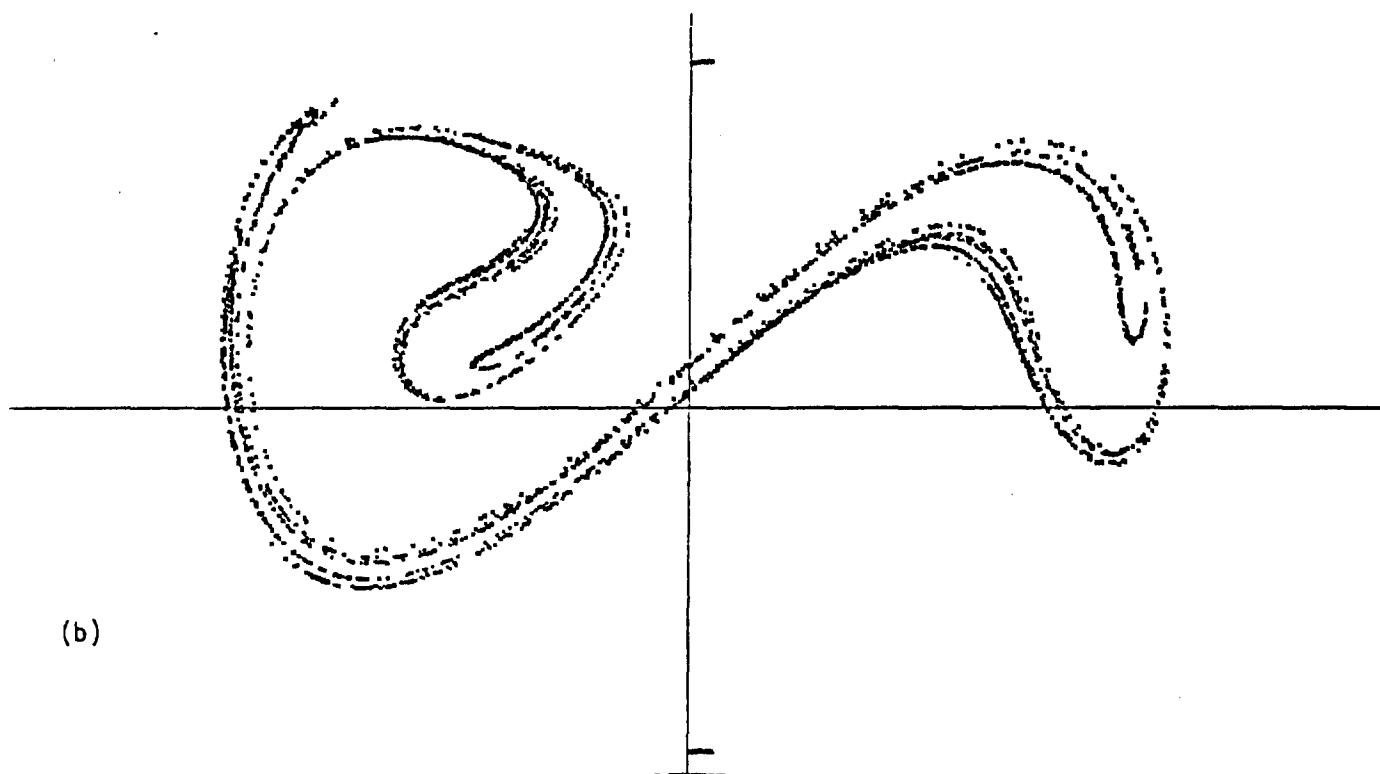


Figure 6.6 (Continued) (c)  $\epsilon\gamma = 0.20$  (inset,  $\epsilon\gamma = 0.19$ ); (d)  $\epsilon\gamma = 0.30$ .



(a)



(b)

Figure 6.7. Numerically computed orbits of the Poincaré map,  $\omega = 1.0$ ,  $\epsilon\gamma = 0.30$ . (a)  $\epsilon\delta = 0.25$ ; (b)  $\epsilon\delta = 0.20$ .



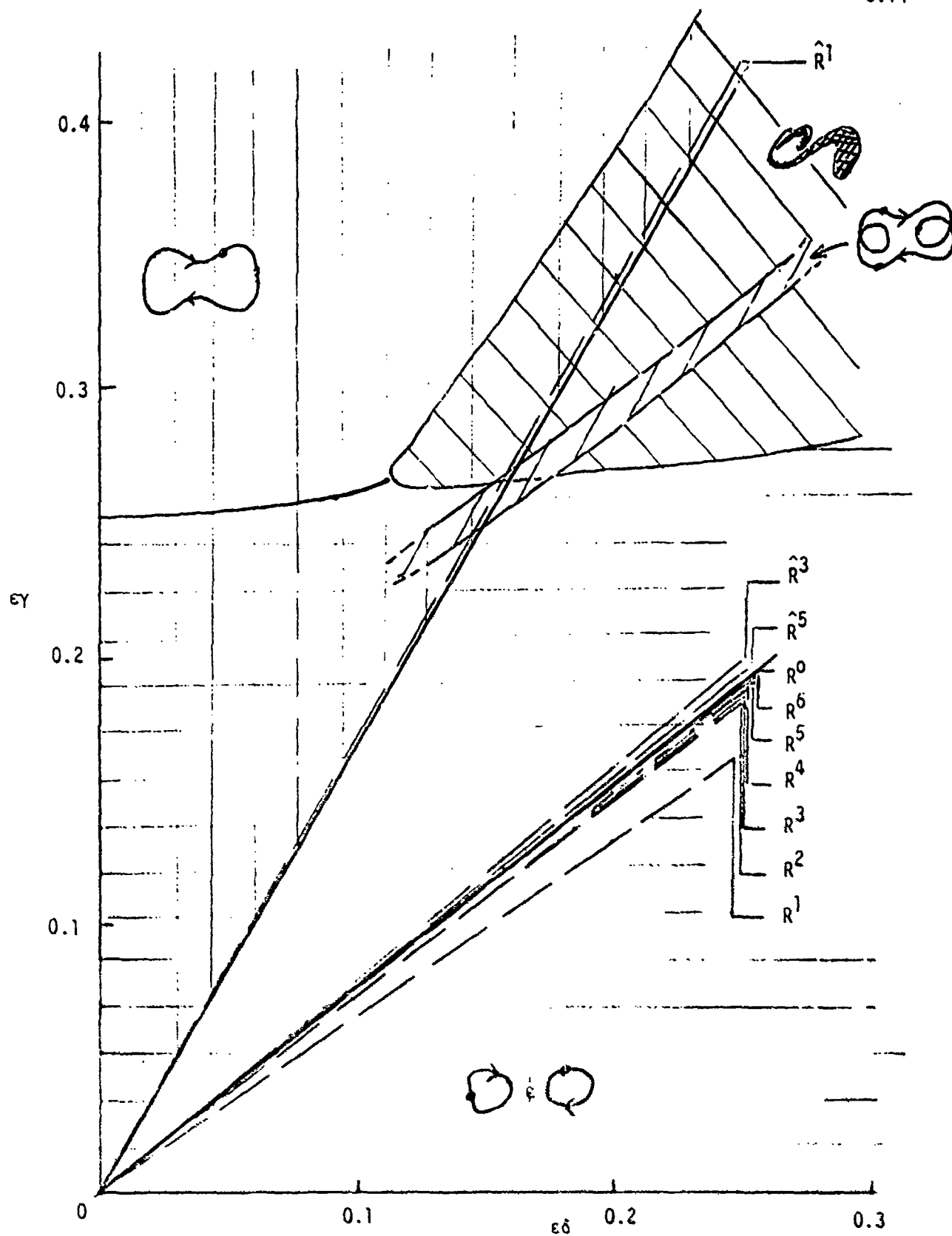


Figure 6.8. Some theoretical and numerical bifurcation curves,  $\omega = 1$ .  
Theoretical subharmonic curves -----, Numerical curves ———.

For small damping ( $\epsilon\delta < 0.1$ ), sustained strange attractor motions are not observed and we find only fixed points (T-periodic motions) or subharmonics of relatively low orders ( $m \leq 5$ ) coexisting with the transient chaos characteristic of horseshoes, cf. Figure 6.4. On the other hand, for higher damping levels we observe stable T-periodic motions outside  $\Gamma_0^+ \cup \{p_0\} \cup \Gamma_-^0$  (cf. Holmes [1979]). One apparently requires moderate forcing and damping for the strange attractor to exist. We note that in some parameter ranges (with multiple shading in Fig. 6.8) two or more attractors (periodic and/or strange) coexist and that one can observe hysteresis in the jumps from one attractor to another as the parameters  $(\epsilon, \gamma, \delta, \omega)$  are slowly varied.

We now attempt to describe the various attracting sets in the light of the theory developed in this article. For low damping the stable subharmonic motions and horseshoes predicted in Sections 3 and 4 account reasonably well for the numerical observations, especially in view of the small characteristic size of the domains of attraction of high subharmonics (eg. (6.9)) and the associated implication that small machine errors will be sufficient to perturb orbits from the stable manifold of one such subharmonic to a neighboring one, leading to a 'transitionally chaotic orbit' (cf. Ueda (1980)). Thus, although there certainly are parameter values for which such long period sinks exist, they are not observable in practice. Franks [1981] has some interesting results on the addition of random perturbations to a motion originally attracted to a stable period 7 sink of the Hénon map (cf. Hénon [1976]). The addition of a small random vector after each iteration leads to an orbit which appears to fill out the same 'strange attractor' as one observes for nearby parameter values without intentional perturbations. This observation is one of the motivations for the conjecture which follows.

The second pertinent observation is also based jointly on analysis and numerical integrations. The study of near resonant excitation of the stable periodic motions near  $(u,v) = (\pm 1, 0)$  in Holmes [1979] and more particularly Holmes and Holmes [1981] shows that for forcing frequencies  $\omega$  near  $\sqrt{2}$  or  $2\sqrt{2}$ , one obtains harmonic and subharmonic bifurcations as  $\gamma$  is increased, so that the stable fixed point of the Poincaré map can become a saddle point and (in the second case) throw off an orbit of period two. As in the case of subharmonics near the homoclinic orbit (cf. §5.2), there is some reason to think that this period doubling bifurcation is the first in a countable sequence which has an accumulation point (cf. Feigenbaum [1978] Collet, Eckmann and Lanford [1980]). The studies of Equation (4.1) and an associated cubic mapping of the plane

$$(u,v) \rightarrow (v, -\beta u + \alpha v - v^3) \quad (6.12)$$

strongly support this idea (Holmes [1979]). In fact in view of the results of Gavrilov and Silnikov outlined in Section 5.2, we can conclude that all the subharmonic sinks of sufficiently high period  $2\pi\omega/\omega$  undergo such period doubling bifurcations as  $\gamma$  increases for fixed  $\delta$ . It seems reasonable to expect these each to be the first of such an accumulating sequence, and all such sequences presumably accumulate on the homoclinic bifurcation. This provides a mechanism by which all sinks of periods less than some fixed integer  $N$  can become unstable, leading to our conjecture:

Conjecture 6.1. For any fixed integer  $N < \infty$  there are open sets of parameter values in  $(\epsilon, \gamma, \delta, \omega)$  space for which the attracting set  $A = \bigcap_{n \geq 0} P^n(A) \equiv C\mathbb{Z}(W^u(p_c))$  of the Duffing equation (4.1) contains a shift on two symbols with a countable set of saddle type periodic orbits and finite sets of periodic sinks, none of the latter's periods being less than  $N$ .

This conjecture implies that  $A$  is not a strange attractor, since it can be decomposed into a finite set of simple periodic attractors. Such an attracting set is structurally stable. We can of course, prove this conjecture for the case  $N = 2$  (Holmes and Holmes [1981]). The conjecture can be proven for arbitrarily high  $N$  using a modified Duffing-van der Pol equation (Holmes [1981]). Numerical work of Chirikov [1979, 1980] and Lieberman [1981] on diffeomorphisms of the plane, also lends support to our conjecture.

Newhouse's results [1974, 1979, 1980] suggest that there are also residual subsets of parameter values for which one has countably many sinks, but a careful examination of the proof of his Theorems 5.4 and 5.5 shows that the small perturbations of diffeomorphisms with quadratic tangencies necessary to obtain such infinite sets of sinks are very special, and there appears to be no guarantee that our specific family realizes these perturbations. In fact some recent work of Guckenheimer [1981] which borrows ideas for Yakobson's [1978] work on one dimensional maps indicates that one can also construct diffeomorphisms close to such homoclinic bifurcations which possess no periodic sinks and therefore presumably have a genuine (indecomposable) strange attractor. (Such an attractor has been proven by Misiurewicz [1980] to exist for the piecewise linear Lozi map. Here the persistent quadratic tangencies and their associated sinks do not occur since the folds in the manifolds become points in the piecewise linear manifolds). On the basis of this work one can conjecture that a situation similar to that in one dimensional quadratic maps holds:

Conjecture 6.2. There is a set of parameter values of positive measure but containing no open sets for which the Duffing equation has a strange attractor  $A = Cl(W^u(p_\epsilon))$ , i.e.  $A$  contains a dense orbit and is indecomposable.

Note that our two conjectures are compatible.

On this note of unbridled speculation, let us close our story.

# APPENDIX. INVARIANT MANIFOLDS AND THE LAMBDA LEMMA

Stable Manifold Theorem. Hirsh-Pugh-Shub [1977]. Suppose that  $p$  is a hyperbolic fixed point of a  $C^r$  diffeomorphism  $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and that  $E^s(p), E^u(p)$  are the stable and unstable eigenspaces of the associated linearized mapping  $DP(p)$ . Then, in a neighborhood  $U(p)$  there are  $C^r$  invariant submanifolds  $W_{loc}^s(p), W_{loc}^u(p)$ , tangent to  $E^s(p), E^u(p)$  at  $p$ . The global manifolds  $W^s(p), W^u(p)$  are injectively immersed copies of  $\mathbb{R}^s, \mathbb{R}^u$ , when  $s = \dim E^s(p), u = \dim E^u(p)$  ( $s + u = n$ ). Moreover, if  $P$  depends smoothly upon parameters  $\mu \in \mathbb{R}^m$ , so do  $W_{loc}^s(p), W_{loc}^u(p)$ .

$\lambda$ -Lemma. (Palis [1969]). Let  $P$  be a  $C^r$  diffeomorphism with a hyperbolic fixed point  $p$  and let  $D^u$  be a  $u$ -disc in  $W^u(p)$ . Let  $\Delta$  be a  $u$ -disc meeting  $W^s(p)$  transversely at some point  $q$ . Then  $\bigcup_{n \geq 0} P^n(\Delta)$  contains  $u$ -discs arbitrarily  $C^r$  close to  $D^u$ .

These results also apply to periodic orbits  $\{p_i\}_{i=1, \dots, n}$   $P^n(p_i) = p_i$  and an analogue of the first result applies to flows, such as that of the suspended system (2.2).

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